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A CONSTRUCTION METHOD FOR POINT DIVISIBLE DESIGNS

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Abstract: In this paper we give a recursive construction method that yields many families of point divisible designs. Under certain conditions we also obtain many strongly divisible 1-designs and 2-designs.

1. Introduction

Adhikary [1] extended the concept of *group divisible* designs, first introduced by Bose and Connor [6], to *generalised group divisible* designs. A *generalised group division* of a 1-design is a partition of the points into classes $\mathscr{P}_1, \ldots, \mathscr{P}_d$, such that the number of blocks through two points depends only on their point classes, and is denoted by λ_{ij}, $(1 \leq i, j \leq d)$.

Independently, the concept of *point divisibility* was introduced in connection with *strong tactical decompositions* of designs (see [2, 3]). A *point division* of a 1-design is a generalised group division such that $\lambda_{ii} = \lambda$ for all i. A *tactical division* of a 1-design is a tactical decomposition whose point classes form a point division. Clearly in the case of a 2-design the terms tactical division and tactical decomposition are synonymous. If the tactical division has c block classes and d point classes, then $b + d \geq v + c$ where b is the number of blocks and v is the number of points (see [3]). Tactical divisions for which $b + d = v + c$ are of special interest and are called *strong*. A 1-design admitting a strong tactical division is said to be *strongly divisible*.

The purpose of this paper is to give a recursive method for the construction of point divisible 1-designs and 2-designs. In particular, the construction yields many strongly divisible 1-designs and 2-designs. For instance, in Section 4, we show that if $\mathscr{H}$ is a Hadamard 2-$(m, \frac{1}{2}(m-1), \frac{1}{4}(m-3)$ design, and $\mathscr{A}$ is an affine plane of order m, then there exists a strongly divisible 2-$(4m^2(t-v)$, $m(2mt - 2mv + t - 2v)$,

$(m+1)(mt-mv+t-2v))$ design whenever there exists a 2-$(4(t-v), t, v)$ symmetric design, of which there are infinitely many.

In Section 5 we use a modified form of the construction process to obtain the affine designs of [9].

2. Basic results and definitions

A *t-structure*, $\mathscr{S}$, is an incidence structure with a finite number of points and blocks, with the properties:

(i) Each block of $\mathscr{S}$ is incident with exactly k points for some integer $k \geqq t$,

(ii) There exists an integer $\lambda > 0$, such that any set of t points is incident with exactly λ blocks.

A *design*, $\mathscr{D}$, is an incidence structure with the properties:

(iii) Whenever blocks x and y are incident with the same set of points, then $x = y$,

(iv) Whenever points P and Q are incident with the same set of blocks, then $P = Q$.

A *t-design* is a t-structure that is also a design. Hence in a t-design we may regard a block as the set of points incident with that block, with incidence as set theoretic inclusion.

If $\mathscr{D}(\mathscr{S})$ contains v points, then we say that $\mathscr{D}(\mathscr{S})$ is a t-(v, k, λ) design (structure). We will denote the number of blocks, and the number of blocks through a point by b and r respectively. (We assume throughout that $b > 1$.)

A *tactical decomposition* of a design (structure) is a partition of the points into classes $\mathscr{P}_1, \ldots, \mathscr{P}_d$ together with a partition of the blocks into classes $\mathscr{B}_1, \ldots, \mathscr{B}_c$, such that for any point class $\mathscr{P}_i$ and any block class $\mathscr{B}_j$, the number of points of $\mathscr{P}_i$ on a block of $\mathscr{B}_j$ depends only on the classes chosen and is denoted by β_{ij}. Dually the number of blocks of $\mathscr{B}_j$ through a point of $\mathscr{P}_i$ depends only on i and j and is denoted by γ_{ij}.

A *parallelism* of a design (structure) is a tactical decomposition with just one point class, such that $\gamma_{ij} = 1$ for all j.

The following result can be found in [2]:

Result 1. If $T(\mathscr{D})$ is a tactical division of a 1-design $\mathscr{D}$, then the following are equivalent:

(i) $T(\mathscr{D})$ is strong;

(ii) The number of points in the intersection of two distinct blocks depends only on their block classes;

(iii) Every two distinct blocks of the same block class intersect in $k - r + \lambda$ points.

A special case of Result 1 is the following result due to Bose [5]:

Result 2. If a 2-design $\mathscr{D}$ has a parallelism, then $b + 1 \geqq v + r$ with equality if and only if $\mathscr{D}$ is affine.

3. The main construction

Let $\mathscr{A}$ be a 2-(v, k, λ) structure with b blocks and r blocks through every point, admitting a parallelism with m blocks in each parallel class, hence $v = mk$. We label the parallel classes $\mathscr{A}_1, \ldots, \mathscr{A}_r$, and let A be an incidence matrix "associated with" this parallelism of $\mathscr{A}$. (If, for a structure $\mathscr{S}$, there is defined a partition of the points (blocks) into d classes of l_i points (blocks) each ($1 \leq i \leq d$), then an incidence matrix associated with this partition is an incidence matrix for $\mathscr{S}$ with the rows (columns) ordered so that the rows (columns) corresponding to the points (blocks) of the nth point (block) class are the $\sum_{i=1}^{n-1} l_i + 1$, $\sum_{i=1}^{n-1} l_i + 2, \ldots, \sum_{i=1}^{n} l_i$ rows (columns) of this matrix.) Then A can be written as $(A_1 A_2 \cdots A_r)$, where A_i is a $v \times m$ incidence matrix of the 1-structure whose points are the points of $\mathscr{A}$ and whose blocks are the blocks of $\mathscr{A}_i$.

Let $\mathscr{S}$ be a 1-(mn, k', r') structure with b' blocks, admitting a *regular point division* (i.e. a point division with all point classes of equal size), with point classes $\mathscr{S}_1, \ldots, \mathscr{S}_n$ and $|\mathscr{S}_i| = m$. We denote the number of blocks mutually incident with a point of $\mathscr{S}_i$ and a point of $\mathscr{S}_j$ by λ_{ij}, and let $\lambda' = \lambda'_{ii}$ for all i. We let S be an incidence matrix for $\mathscr{S}$ associated with this point division. Let

$$B = [I_n \otimes A_1 \quad I_n \otimes A_2 \cdots I_n \otimes A_r],$$

where I_n is the $n \times n$ identity matrix, and $\otimes$ denotes Kronecker multiplication. Then B is a $vn \times mnr$ matrix. We point out that, essentially, we have taken n isomorphic copies of $\mathscr{A}$.

Let $C = I_r \otimes S$. Thus C is an $mnr \times b'r$ matrix.

Finally let $D = BC$.

Theorem 1. *D is the incidence matrix of a* 1-(mnk, kk', rr') *structure $\mathscr{D}$, admitting a regular point division with n point classes.*

Proof. Since every row of $I_n \otimes A_i$ has exactly one entry $+1$, it is clear that D is a $(0, 1)$ matrix. Further, since every column of $I_n \otimes A_i$ has exactly k entries $+1$ and every column of S has k' entries $+1$, we see that every column of D has kk' entries $+1$. Thus D is the incidence matrix of a structure $\mathscr{D}$ with constant block size kk'.

We now consider $DD^{\mathrm{T}} = BCC^{\mathrm{T}}B$. Clearly $CC^{\mathrm{T}} = I_r \otimes SS^{\mathrm{T}}$. However, since S has a regular point division,

$$SS^{\mathrm{T}} = (r' - \lambda')I_{mn} + \Lambda,$$

where

$$\Lambda = (I_n \otimes j_m^{\mathrm{T}})\bar{\Lambda}(I_n \otimes j_m)$$

and j_m is the all $+1$ vector of length m, and $\bar{\Lambda} = (\lambda'_{ij})$. Hence

$$CC^{\mathrm{T}} = (r' - \lambda')I_{mnr} + I_r \otimes \Lambda.$$

Thus

$$DD^T = (r' - \lambda')BB^T + B(I_r \otimes \Lambda)B^T.$$

Since $\mathscr{A}$ is a 2-structure it is easily verified that

$$BB^T = (r - \lambda)I_{mnk} + (I_n \otimes \lambda J_{mk}),$$

where J_{mk} is the $mk \times mk$ matrix every entry of which is $+1$. Computation of $B(I_r \otimes \Lambda)B^T$ yields

$$r(I_n \otimes j_m^T)\bar{\Lambda}(I_n \otimes j_m).$$

Hence

$$DD^T = (r' - \lambda')(r - \lambda)I_{mnk} + D'$$

where

$$D' = \begin{bmatrix} [(r'-\lambda')\lambda + r\lambda']J_{mk} & r\lambda'_{12}J_{mk} & \cdots & r\lambda'_{1n}J_{mk} \\ r\lambda'_{21}J_{mk} & [(r'-\lambda')\lambda + r\lambda']J_{mk} & \cdots & r\lambda'_{2n}J_{mk} \\ \vdots & \vdots & \ddots & \vdots \\ r\lambda'_{n1}J_{mk} & r\lambda'_{n2}J_{mk} & \cdots & [(r'-\lambda')\lambda + r\lambda']J_{mk} \end{bmatrix}. \quad (*)$$

However, the entries of DD^T are just the inner products of rows of D, i.e. they count the number of blocks common to two points of $\mathscr{D}$. Thus we see that D has a "natural" regular point division with n point classes and mk points in a point class. The point classes are in $(1-1)$ correspondence with the points of the isomorphic copies of $\mathscr{A}$.

Clearly $\mathscr{D}$ is not necessarily a design, since it may have blocks that are identical as point sets or, dually, points incident with the same set of blocks. We let $\mu_{\mathscr{A}}$ denote the greatest intersection number of $\mathscr{A}$.

Lemma 1. *$\mathscr{D}$ is a design if:*

(i) $k > \mu_{\mathscr{A}}(m-1)$

(ii) *$\mathscr{S}$ is a design and no block of $\mathscr{S}$ contains all the points from any given point class of $\mathscr{S}$.*

Proof. We label the first b' columns of D the first column class, the next b' columns the second column class, and so on.

A block of $\mathscr{D}$ is obtained by taking the union of k' blocks of $\mathscr{A}$ or its isomorphic copies. Since the blocks of each parallel class of $\mathscr{A}$ are disjoint, we can never obtain identical (as point sets) blocks within a column class of $\mathscr{D}$ provided $\mathscr{S}$ is a design. Suppose y_1 and y_2 are two blocks of $\mathscr{D}$ (necessarily from different column classes) that are identical as point sets. Let x be a block of either $\mathscr{A}$ or one of its isomorphic copies which is contained in y_1. Then, by (ii), y_2 contains at most $m-1$ blocks from any parallel class of $\mathscr{A}$ or one of its isomorphic copies.

(Since y_1 and y_2 belong to different column classes y_2 cannot contain any block from the parallel class of x.) But, since $k>\mu_{\mathscr{A}}(m-1)$, y_2 cannot contain all k points of x, and thus we have a contradiction.

To show that two points of $\mathscr{D}$ are never incident with the same set of blocks we need only show that the off-diagonal entries of DD^{T} are always less than rr'. But $r\lambda'_{ij}<rr'$ since $\mathscr{S}$ is a design, and

$$(r'-\lambda')\lambda+r\lambda'=(r-\lambda)(\lambda'-r')+rr'<rr',$$

since $r>\lambda$ and $r'>\lambda'$.

Remark 1. Condition (i) implies $\mathscr{A}$ is a deisgn.

Remark 2. If $\mathscr{A}$ is an affine design, then its parameters may be written 2-$(\mu m^2, \mu m, (\mu m-1)/(m-1))$, and thus, since $\mu=\mu_{\mathscr{A}}$, condition (i) is automatically satisfied.

Lemma 2. *$\mathscr{D}$ is a 2-structure if and only if*

$$\lambda r'+(r-\lambda)\lambda'=r\lambda'_{ij} \quad \text{for all } i,j: 1\leq i,j\leq n, i\neq j.$$

Proof. $\mathscr{D}$ is a 2-structure if and only if all the non-diagonal elements of DD^{T} are equal. Hence (*) gives the result.

Remark. If $\mathscr{D}$ is a 2-structure we see that $\mathscr{S}$ must be a group divisible design, i.e. we must have $\lambda'_{ij}=\lambda'_1$ for all i and j.

Lemma 3. *If $\mathscr{S}$ admits a point regular tactical division (i.e. a tactical decomposition whose point classes form a regular point division) $T(\mathscr{S})$ with point classes $\mathscr{S}_1,\ldots,\mathscr{S}_n$ and block classes $\mathscr{C}_1,\ldots,\mathscr{C}_c$, then $\mathscr{D}$ admits a point regular tactical division $T(\mathscr{D})$ with n point classes and rc block classes.*

Proof. Let γ_{ij} denote the number of blocks of $\mathscr{C}_j$ incident with a point of $\mathscr{S}_i$, and let β_{ij} denote the number of points of $\mathscr{S}_i$ incident with a block of $\mathscr{C}_j$. Further suppose $|\mathscr{C}_j|=m_j$, $1\leq j\leq c$.

Using the notation of Theorem 1 we assume that S is an incidence matrix for $\mathscr{S}$ associated with $T(\mathscr{S})$. We call the points indexed by the first mk rows of D, $\mathscr{P}_1$, the points indexed by the next mk rows, $\mathscr{P}_2$, and so on to $\mathscr{P}_n$. We call the blocks indexed by the first m_1 columns of D, $\mathscr{Y}_1$, the blocks indexed by the next m_2 columns of D, $\mathscr{Y}_2$, and so on to $\mathscr{Y}_c$, the blocks indexed by the next m_1 columns of D, $\mathscr{Y}_{c+1}$, and so on to $\mathscr{Y}_{rc}$. We shall show that these point and block classes define a tactical decomposition of $\mathscr{D}$, and hence, by Theorem 1, a point regular tactical division.

Without loss of generality, consider the $mk\times m_j$ submatrix of D whose rows are indexed by the points of $\mathscr{P}_i$ $(1\leq i\leq n)$, and whose columns are indexed by the blocks of

$$\mathcal{Y}_{tc+j} \quad (0 \leqq t \leqq r-1; 1 \leqq j \leqq c).$$

The entries of this submatrix are obtained by multiplying the $(i-1)mk+1, \ldots, imk$ rows of B by the

$$tb' + \sum_{l=1}^{j-1} m_l, \ldots, tb' + \sum_{l=1}^{j} m_l$$

columns of C. However, since each of these rows of B has exactly one entry of $+1$ in the positions $tmn+(i-1)m+1, \ldots, tmn+im$, the row sum for each row of this submatrix will be exactly the row sum for each row of the corresponding $m \times m_j$ submatrix of C containing rows $tmn+(i-1)m+1, \ldots, tmn+im$ and columns

$$tb' + \sum_{l=1}^{j-1} m_l + 1, \ldots, tb' + \sum_{l=1}^{j} m_l.$$

However, by our hypothesis these row sums are all γ_{ij}. Thus the $mk \times m_j$ submatrix of D has all its row sums γ_{ij}.

By similar considerations we can show that this submatrix has all its column sums $k\beta_{ij}$.

Thus $\mathcal{D}$ admits a tactical decomposition with n point classes and rc block classes.

Lemma 4. *$T(\mathcal{D})$ is strong if and only if $b'r+n=mnk+rc$.*

Proof. $T(\mathcal{D})$ is strong if and only if "$b+d=v+c$", i.e. if and only if $b'r+n=mnk+rc$.

Theorem 2. *Any two of the following conditions imply the third:*
(i) *$\mathcal{A}$ is affine;*
(ii) *$T(\mathcal{S})$ is strong;*
(iii) *$T(\mathcal{D})$ is strong.*

Proof. Suppose (i) holds, i.e. suppose $\mathcal{A}$ is a 2-$(\mu m^2, \mu m, (\mu m-1)/(m-1))$ design with $m(\mu m^2-1)/(m-1)$ blocks and $(\mu m^2-1)/(m-1)$ blocks through a point.

$$\begin{aligned} T(\mathcal{S}) \text{ is strong} \quad & \text{if and only if} \quad b'+n = mn+c; \\ & \text{if and only if} \quad rb'-rc = \frac{(\mu m^2-1)}{(m-1)}mn - \frac{(\mu m^2-1)}{(m-1)}n \\ & \qquad = (\mu m^2-1)n \\ & \qquad = mnk-n; \\ & \text{if and only if} \quad T(\mathcal{D}) \text{ is strong} \quad \text{(by Lemma 4).} \end{aligned}$$

Hence (i) and (ii) hold if and only if (i) and (iii) hold. Suppose (ii) holds, i.e. suppose $b'+n=mn+c$.

$T(\mathscr{D})$ is strong if and only if $b'r+n=mnk+rc$ (by Lemma 4);
if and only if $(mn-n)r=mnk-n$ (since $T(S)$ is strong);
if and only if $b-r=v-1$;
if and only if (i) holds (by Result 2).

Hence (ii) and (iii) hold if and only if (i) and (ii) hold.

We now turn our attention to the intersection numbers of $\mathscr{D}$. If $\mathscr{S}$ is strongly divisible, then the number of points in the intersection of a block of class $\mathscr{C}_i$ and a block of class $\mathscr{C}_j$ depends only on i and j. We denote this number by ρ'_{ij}. Further, $\rho'_{ii}=\rho'=k'-r'+\lambda'$ for all i (see Result 1). In the case when $\mathscr{D}$ admits a strong tactical division with block classes $\mathscr{Y}_1,\ldots,\mathscr{Y}_{rc}$ (employing the notation of Lemma 3), we denote the number of points in the intersection of a block from $\mathscr{Y}_i$ and a block from $\mathscr{Y}_j$ by ρ_{ij} $(1\leqq i,j\leqq rc)$ and denote the number of points in the intersection of two blocks of the same block class by ρ.

Theorem 3. *If $\mathscr{A}$ is an affine design and $\mathscr{S}$ admits a strong tactical division $T(\mathscr{S})$, then $\mathscr{D}$ admits a strong tactical division $T(\mathscr{D})$ and its intersection numbers are:*

$$\rho_{ij}=k\rho'_{uw}\quad 1\leqq i,j\leqq rc,\quad i\not\equiv j(\mathrm{mod}\, c)\quad 1\leqq u,w\leqq c, u\equiv i(\mathrm{mod}\, c), w\equiv j(\mathrm{mod}\, c);$$

$$\rho_{ij}=\left[\rho'+\frac{(r'-\lambda')}{m_t}\right]k\quad 1\leqq i,j\leqq rc, i\equiv j\equiv t(\mathrm{mod}\, c), i\neq j, 1\leqq t\leqq c;\quad \rho=k\rho'.$$

In order to prove Theorem 3 we need the following result from [3].

Result 3. Let $T(\bar{\mathscr{D}})$ be a strong tactical division of a 1-design $\bar{\mathscr{D}}$ with $\bar{c}$ block classes and $\bar{d}$ point classes. Then

(i) $\bar{m}_j\bar{\beta}_{ij}=\bar{l}_i\bar{\gamma}_{ij}$ for all $i,j: 1\leqq i\leqq\bar{c}; 1\leqq j\leqq\bar{d}$;

(ii) $\displaystyle\sum_{i=1}^{d}\frac{\bar{\beta}_{ij}\bar{\beta}_{ik}}{\bar{l}_i}=\bar{\rho}_{jk}+\frac{(\bar{r}-\bar{\lambda})\delta}{\bar{m}_j}$ for all $j,k: 1\leqq j,k\leqq\bar{c}$;

where $\delta=1$ if $j=k$ and $\delta=0$ otherwise; and $\bar{l}_i$ is the size of the ith point class. (The remainder of the notation is analogous to that used throughout.)

Proof of Theorem 3. This computation is somewhat lengthy but nevertheless straightforward. We calculate $D^{\mathrm{T}}D$ using techniques similar to those used in Theorem 1. Since each entry in $D^{\mathrm{T}}D$ is just the inner product of two columns of D, the entries of $D^{\mathrm{T}}D$ are the intersection numbers of $\mathscr{D}$. Application of Result 3(ii) yields the result.

Remark. By Theorem 3, if $\mathscr{S}$ has i intersection numbers, then $\mathscr{D}$ has at most $i+c$ intersection numbers.

4. An application of the construction

In this section we give an application of the construction which yields not only many infinite families of point divisible and strongly divisible 1-designs, but also many infinite families of strongly divisible 2-designs.

We begin by constructing a strongly group divisible design $\mathscr{S}$ (i.e. a design admitting a strong tactical division the point classes of which form a group division) using a method which is a generalization on the construction of Sillitto [11].

Let $\mathscr{M}$ be a 2-(m,h,λ) design with b blocks and r blocks through every point, and let $\mathscr{N}$ be a 2-(n,t,v) design with $\bar{b}$ blocks and $\bar{r}$ blocks through every point. (We include the possibility that either $\mathscr{M}$ or $\mathscr{N}$ may be trivial 2-designs, i.e. every set of h (respectively t) distinct points being a block). Let M and N be incidence matrices for $\mathscr{M}$ and $\mathscr{N}$ respectively. Let

$$S=N\otimes M+(J-N)\otimes(J-M).$$

Lemma 5. *S is an incidence matrix of a*

$$1\text{-}(mn, ht+(m-h)(n-t), r\bar{r}+(b-r)(\bar{b}-\bar{r}))$$

design, $\mathscr{S}$, admitting a tactical decomposition $T(\mathscr{S})$ with n point classes and $\bar{b}$ block classes. Further, no block of $\mathscr{S}$ contains all the points of a point class of $\mathscr{S}$.

Proof. We let the ith point class of $T(\mathscr{S})$ be the points indexed by the $(i-1)m+1,\ldots, im$ rows of $S(1\leq i\leq n)$, and the jth block class of $T(\mathscr{S})$ be the blocks indexed by the $(j-1)b+1,\ldots, jb$ columns of $S(1\leq j\leq \bar{b})$. Since $\mathscr{N}$ and $\mathscr{M}$ are both 2-designs, and, as above we assume $\bar{b}>1$, the result follows immediately.

Lemma 6. *$T(\mathscr{S})$ is a tactical group division if and only if $\bar{b}=4(\bar{r}-v)$.*

Proof. Computation yields:

$$\begin{aligned}SS^{\mathrm{T}}=I_n\otimes&\{[(r-\lambda)\bar{b}]I_m+[\lambda\bar{r}+(b-2r+\lambda)(\bar{b}-\bar{r})]J_m\}\\&+(J_n-I_n)\otimes\{[(r-\lambda)(\bar{b}-4\bar{r}+4v)]I_m\\&+[\lambda v+(b-2r+\lambda)(\bar{b}-2\bar{r}+v)+2(r-\lambda)(\bar{r}-v)]J_m\}.\end{aligned}$$

Clearly the point classes of $T(\mathscr{S})$ form a group division if and only if $(r-\lambda)(\bar{b}-4\bar{r}+4v)=0$, i.e. $\bar{b}=4(\bar{r}-v)$.

Lemma 7. *$\mathcal{S}$ is a 2-design if and only if $\bar{b}=4(\bar{r}-v)$ and $b=4(r-\lambda)$* (*cf.* [10], [11]).

Proof. Clearly $\mathcal{S}$ is a 2-design if and only if $\bar{b}=4(\bar{r}-v)$ and

$$\lambda\bar{r}+(b-2r+\lambda)(\bar{b}-\bar{r})=\lambda v+(b-2r+\lambda)(\bar{b}-2\bar{r}+v)$$
$$+2(r-\lambda)(\bar{r}-v), \quad \text{i.e. } b=4(r-\lambda).$$

Lemma 8. *$T(\mathcal{S})$ is strong if and only if $\mathcal{N}$ and $\mathcal{M}$ are both symmetric.*

Proof. $T(\mathcal{S})$ is strong if and only if "$b+d=v+c$", i.e. if and only if $b\bar{b}+n=mn+\bar{b}$. However since $\bar{b}\geq n$ and $b\geq m$, we have $T(\mathcal{S})$ strong if and only if $\bar{b}=n$ and $b=m$.

Lemma 9. *There exist infinitely many symmetric 2-designs with "$b=4(r-\lambda)$".*

Proof. If $\mathcal{M}$ and $\mathcal{N}$ are both symmetric 2-designs with "$b=4(r-\lambda)$", then $\mathcal{S}$ is a symmetric 2-design with "$b=4(r-\lambda)$". Since there exists both a 2-(4,3,2) design and a 2-(16,6,2) design with the required properties we can use this construction method recursively to obtain an infinite family.

Lemma 10. *Let $\mathcal{N}$ be a symmetric 2-(n,t,v) design with $n=4(t-v)$, and let $\mathcal{M}$ be a symmetric 2-(m,h,λ) design. Then $\mathcal{S}$ is a strongly group divisible design with parameters:*

$$b=v=4(t-v)m; \qquad c=d=4(t-v); \qquad r=k=3mt+4hv-2ht-4mv;$$
$$\rho'=\lambda'=3mt-4mv-6ht+8hv+4\lambda t-4\lambda v;$$
$$\rho'_{ij}=\lambda'_{ij}=2mt-3mv-2ht+4hv \quad 1\leq i,j\leq n, i\neq j.$$

Proof. This follows immediately from Lemmas 5, 6 and 8, and observing that $SS^{\mathrm{T}}=S^{\mathrm{T}}S$.

Theorem 4. *Let $\mathcal{A}$ be a 2-$(\mu m^2, \mu m, (\mu m-1)/(m-1))$ affine design (if such a design exists), and let $\mathcal{S}$ be the strongly group divisible 1-design of Lemma 10 above. Then $\mathcal{D}$ as constructed in Theorem 1 is a strongly divisible 1-design, and is a 2-design if and only if $\mu=1$ and $\mathcal{M}$ is a 2-$(4\lambda+3,\ 2\lambda+1,\ \lambda)$ Hadamard design or its complement.*

Proof. By Theorems 1 and 2 and Lemmas 1 and 3, $\mathcal{D}$ is clearly a strongly divisible 1-design. By Lemma 2, $\mathcal{D}$ is a 2-design if and only if

$$\left(\frac{\mu m-1}{m-1}\right)(3mt+4hv-2ht-4mv)$$
$$+\mu m(3mt-4mv-6ht+8hv+4\lambda t-4\lambda v)$$
$$=\left(\frac{\mu m^2-1}{m-1}\right)(2mt-3mv-2ht+4hv).$$

Since $\lambda(m-1)=h(h-1)$, on substitution and dividing by $m(t-v)$ we obtain:

$$\mu[(2h-1)\lambda-h(h-1)]^2=\lambda^2 \qquad (1)$$

If $\mu>1$, then

$$[(2h-1)\lambda-h(h-1)]^2<\lambda^2, \quad \text{i.e.}\left(\lambda-\frac{h-1}{2}\right)\left(\lambda-\frac{h}{2}\right)<0, \quad \text{i.e.}\ \frac{h}{2}>\lambda>\frac{h-1}{2};$$

which is clearly not possible since h and λ are integers. Thus the only solutions of (1) are $\mu=1$ and $h=2\lambda+1$ or $h=2\lambda$, which implies that $\mathscr{M}$ is a symmetric Hadamard design or its complement.

Theorem 5. *Let $\mathscr{M}$ be a 2-$(4\lambda+3, 2\lambda+1, \lambda)$ Hadamard design such that there exists an affine plane $\mathscr{A}$ of order $m=4\lambda+3$. Then there exists an infinite family of strongly divisible*

$$2\text{-}(4m^2(t-v), m(2mt-2mv+t-2v), (m+1)(mt-mv+t-2v))$$

designs with $4(t-v)$ point classes and intersection numbers:

$$m(mt-mv-v),\ m(mt-mv-2v+t) \qquad \text{and} \qquad (m+1)^2(t-v)-mt.$$

Proof. Let $\mathscr{N}$ be any one of the infintely many symmetric 2-$(4(t-v), t, v)$ designs that exist by Lemma 9. Then $\mathscr{S}$ as constructed in Lemma 5 is a strongly divisible 1-design (by Lemma 10), with parameters:

$$\begin{aligned} &v=b=4m(t-v); \qquad c=d=4(t-v);\\ &r=k=2mt-2mv+t-2v; \qquad\qquad (2)\\ &\rho'=\lambda'=mt-mv-v;\\ &\rho'_{ij}=\lambda'_{ij}=mt-mv+t-2v: \quad 1\leq i,j\leq 4(t-v), i\neq j. \end{aligned}$$

Since there exists an affine plane $\mathscr{A}$ of order m, we can construct $\mathscr{D}$ as in Theorem 1 using $\mathscr{S}$ and $\mathscr{A}$, taking $n=4(t-v)$. By Lemmas 1, 2 and 3 and Theorems 1 and 3 we see that $\mathscr{D}$ has the required properties.

Remark 1. Whenever m is a prime power it is known that there exists an affine plane of that order, and many 2-$(4\lambda+3,\ 2\lambda+1,\ \lambda)$ Hadamard designs with $m=4\lambda+3$ a prime power are known to exist

Remark 2. We point out that the strongly divisible 2-designs constructed above have three intersection numbers: "$k-r+\lambda$", "$\lambda v/b$" and "$(r-\lambda)/|\mathscr{B}|+k-r+\lambda$" where $|\mathscr{B}|$ is the size of a block class in a regular maximal decomposition. Thus the strongly regular graph of such a design (obtained by taking the vertices of the graph to be the block classes of the decomposition of the design, with two vertices adjacent if and only if a block from each of the corresponding block classes intersect in x points where x is one of the intersection numbers apart from $k-r+\lambda$) is complete bipartite or its complement. For further details see [4].

Remark 3. Given any strongly divisible 1-design with the parameters of the $\mathscr{S}$ of Theorem 5, we can construct a strongly divisible 2-design; i.e. it is not necessary for the construction of Theorem 1 that $\mathscr{S}$ belongs to the class of designs constructed in Theorem 5. For instance, the authors have constructed a strongly divisible 1-(16,7,7) design $\mathscr{S}$ (i.e. taking $m=4$, $t=3$, $v=2$ in (2)), and since there exists an affine plane of order 4, this can be used to construct a strongly divisible 2-(64,28,15) design. This strongly divisible 1-(16,7,7) design, however, is not a member of the family of designs of Theorem 5.

We point out that in a recent paper by John and Turner [8], the strongly divisible 1-(16,7,7) mentioned above and also a strongly divisible 1-(20,9,9) have been found using a computer aided search. Since there exists an affine plane of order S we may apply our construction to the latter of these designs to obtain a strongly divisible 2-(100,45,24) design.

5. A modification of the construction

It is clear that the construction of Section 3 can be modified and generalised in many ways, and below we give an example of how a modified form of this construction can be used to obtain the affine designs of Kimberley [9].

Construct $\mathscr{D}$ with matrix D from $\mathscr{A}$ and $\mathscr{S}$ as in Section 3, with the extra condition that every block of $\mathscr{S}$ is incident with precisely m points of $\mathscr{S}$ (i.e. $k=m$). Then, by Theorem 1, D is the incidence matrix of a 1-(mnk, mk, rr') structure $\mathscr{D}$, admitting a regular point division with n point classes.

Let $\Delta = I_n \otimes j_{mk}^{\mathrm{T}}$. Then Δ is an $mnk \times m$ matrix, and every column of Δ contains precisely mk entries of $+1$. Let $\bar{D} = (D\Delta)$.

Theorem 6. *$\bar{D}$ is the incidence matrix of a* 1-(mnk, mk, $rr'+1$) *structure $\bar{\mathscr{D}}$, admitting a regular point division with n point classes.*

Proof. The proof is identical to that of Theorem 1, with the exception that $\bar{D}\bar{D}^{\mathrm{T}} = DD^{\mathrm{T}} + \Delta\Delta^{\mathrm{T}}$. Clearly $\Delta\Delta^{\mathrm{T}} = I_n \otimes J_{mk}$.

So

$$\bar{D}\bar{D}^{\mathrm{T}} = (r'-\lambda')(r-\lambda)I_{mnk} + D' + I_n \otimes J_{mk}$$

where D' is given by (*) above. So, as in Theorem 1, we have a "natural" point division.

Lemma 11. *$\bar{\mathscr{D}}$ is a design if the conditions of Lemma 1 are satisfied.*

Proof. Since $\mathscr{D}$ is a design, by Lemma 1, and since no two columns of Δ are the same, we need only observe that, since no block of $\mathscr{S}$ contains all the points of a point class of $\mathscr{S}$, no block of $\mathscr{D}$ contains all the points of a point class of the "natural" point division of $\mathscr{D}$. Hence $\bar{\mathscr{D}}$ is a design.

Lemma 12. *$\bar{\mathscr{D}}$ is a 2-structure if and only if*

$$\lambda r' + (r-\lambda)\lambda' + 1 = r\lambda'_{ij} \quad \text{for all } i,j: 1 \leqq i,j \leqq n, i \neq j.$$

Proof. cf. Lemma 2.

Lemma 13. *If $\bar{\mathscr{D}}$ is a 2-structure and $\mathscr{S}$ admits a point regular tactical division $T(\mathscr{S})$ with point classes $\mathscr{S}_1, \ldots, \mathscr{S}_n$ and block classes $\mathscr{C}_1, \ldots, \mathscr{C}_c$, such that the number of blocks of $\mathscr{C}_j$ incident with a point of $\mathscr{S}_i$ depends only on the block class $\mathscr{C}_j$, (i.e. there exists a γ_j for all j $(1 \leqq j \leqq c)$ such that $\gamma_{ij} = \gamma_j$ for all i $(1 \leqq i \leqq n)$—where γ_{ij} is as in Lemma 3), then $\bar{\mathscr{D}}$ admits a tactical division $R(\bar{\mathscr{D}})$ with one point class and rc + 1 block classes.*

Proof. Let the block classes of $R(\bar{\mathscr{D}})$ be the block classes of $T(\mathscr{D})$ as given in Lemma 3, with the blocks corresponding to the columns of Δ as one extra block class. The proof then follows by Lemma 3.

Remark. If $T(\mathscr{S})$ is strong we may replace the condition $\gamma_{ij} = \gamma_j$ in the above, by the condition

$$m_j = \frac{n(r'-\lambda')}{(m-\rho' n)}, \quad 1 \leqq j \leqq c.$$

Since

$$\sum_{i=1}^{n} \gamma_{ij} = \sum_{i=1}^{n} \frac{m_j \beta_{ij}}{l_i} \qquad \text{(by Result 3(i))}$$

$$= m_j k'/m = m_j \qquad (\text{since } k' = m).$$

Let $\gamma = m_j/n$

$$\sum_{i=1}^{n} \gamma_{ij}^2 = \sum_{i=1}^{n} \frac{m_j^2 \beta_{ij}^2}{l_i^2} \qquad \text{(Result 3(i))}$$

$$= \frac{m_j}{m}(\rho' m_j + r' - \lambda') \qquad \text{(Result 3(ii))}.$$

So $\gamma_{ij} = \gamma_j$ for all i if and only if

$$\sum_{i=1}^{n} (\gamma_{ij} - \gamma)^2 = 0 \quad \text{for all } j;$$

i.e.

$$\frac{m_j}{m}(\rho' m_j + r' - \lambda') - \frac{m_j^2}{n} = 0 \quad \text{for all } j;$$

i.e.

$$m_j = \frac{n(r'-\lambda')}{(m-n\rho')} \quad \text{for all } j,\ 1 \leq j \leq c.$$

Lemma 4. *$R(\mathscr{D})$ is strong if and only if $b'r+n=mnk+rc$.*

Proof. $R(\mathscr{D})$ is strong if and only if "$b+1=v+c$", i.e. if and only if $(b'r+n)+1 = mnk+(rc+1)$.

Theorem 7. *If $\mathscr{D}$ and $\mathscr{S}$ satisfy the conditions of Lemma 13, then any two of the following imply the third:*

(i) *$\mathscr{A}$ is affine;*
(ii) *$T(\mathscr{S})$ is strong;*
(iii) *$R(\mathscr{D})$ is strong.*

Proof. cf. Theorem 2.

We now construct the designs of [9] by this method.

Theorem 8. *Let $\mathscr{A}$ be an affine 2-$(\mu m^2, \mu m, (\mu m-1)/(m-1))$ design, and let $\mathscr{S}'$ be an affine plane of order m. Suppose the parallel classes of $\mathscr{S}'$ are $\mathscr{S}_1,\ldots,\mathscr{S}_{m+1}$. Then if $\mathscr{S}$ is the incidence structure whose points are the points of $\mathscr{S}'$, and whose blocks are the blocks of $\mathscr{S}_1,\ldots,\mathscr{S}_m$ with incidence as in $\mathscr{S}'$, then $\mathscr{D}$, constructed as in Theorem 1 using $\mathscr{S}$ and $\mathscr{A}$ is an affine 2-$(\mu m^3, \mu m^2, (\mu m^2-1)/(m-1))$ design.*

Proof. $\mathscr{S}$ is clearly a 1-(m^2, m, m) design. Define $T(\mathscr{S})$ as follows: Let the block classes of $T(\mathscr{S})$ be the parallel classes $\mathscr{S}_1,\ldots,\mathscr{S}_m$ of $\mathscr{S}'$. Let the point classes of $T(\mathscr{S})$ be the point sets of the blocks of $\mathscr{S}_{m+1}$. Then $T(\mathscr{S})$ has m block (point) classes of size m. Every block (point) of $\mathscr{S}$ is incident with precisely one point (block) of any given point (block) class. So $T(\mathscr{S})$ is a strong tactical decomposition satisfying the condition of Lemma 13. Any two points of $\mathscr{S}$ are on 0 or 1 common blocks depending only on whether they are from the same or different point classes respectively, and so $T(\mathscr{S})$ is a strong tactical division. So, by Theorems 6, 7 and Lemmas 11, 12 and 13, $\mathscr{D}$ is a 2-$(\mu m^3, \mu m^2, (\mu m^2-1)/(m-1))$ design admitting a strong tactical decomposition $R(\mathscr{D})$ with one point class. By Lemma 3, the number of blocks of any block class (apart from the one derived from Δ) incident with any point is $\gamma_{ij}=1$. Also, every row of Δ contains precisely one $+1$. So $R(\mathscr{D})$ is a parallelism. Hence, by Result 2, $\mathscr{D}$ is affine.

6. General remarks

In our examples of the use of the construction of Section 3 we have mainly restricted ourselves to cases in which the structure $\mathscr{D}$ obtained is a 2-design. Clearly this construction process can yield many more point divisible 1-designs. For example, we could use the

$$2\text{-}(4m^2(t-v),\ m(2mt+t-2mv-2v),\ (m+1)(mt-mv+2v-t))$$

designs constructed in Section 4 in conjunction with an affine plane of order m^2 to obtain further group divisible 1-designs, utilising the recursive nature of this construction.

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