

# Another Postage Stamp Problem

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The postage stamp problem requires the selection of a set of  $k$  postage stamp denominations such that sums of  $h$  (or fewer) of these denominations can realise all the numbers  $1, 2, \dots, n$  for  $n$  as large as possible (given  $k$ ). In this paper we consider a natural analogue of this problem to the case where each stamp denomination is allowed to occur a negative number of times, so long as the sum of the absolute values of the numbers of occurrences is at most  $h$ . Interestingly, for the case  $h = 2$ , the problem considered here is also an analogue of the Golomb Ruler problem. For the case  $h = 2$ , constructions are shown which give a lower bound on the maximum  $n$  achievable.

Received August 1987, revised January 1988

## 1. THE POSTAGE STAMP PROBLEM AND ITS ANALOGUE

The Postage Stamp problem has been studied for many years under a variety of guises, and can be stated very simply. It requires the selection, for given positive integers  $h$  and  $k$ , of a set  $A$  of  $k$  positive integers, or 'postage stamp denominations', such that:

(i) sums with  $h$  (or fewer) terms, where each term is equal to an integer in  $A$ , can realise all the numbers  $1, 2, \dots, n$ ;

(ii)  $n$  is as large as possible.

Clearly,  $A$  must have smallest element 1. We call  $A$  a  $(k, n, h)$ -Postage Stamp Set, or  $(k, n, h)$ -PSS.

The problem can be informally stated as that of choosing a set of  $k$  postage stamp denominations so that every value between 1 and  $n$  can be achieved using at most  $h$  stamps. This has given rise to the name of the problem. A  $(k, n, h)$ -PSS with maximal  $n$  for given  $k$  and  $h$  is called an *Extremal* PSS, and the value of  $n$  obtained is denoted by  $n_h(k)$ .

A number of authors have studied the problem of determining the maximum  $n$  that can be obtained for any given pair  $(k, h)$ . Tables of the best known values of  $n$  for some small  $k$  and  $h$  can be found in the papers of Alter and Barnett,<sup>1</sup> Lunnon<sup>9</sup> and Mossige.<sup>10, 11</sup> The papers of Alter and Barnett<sup>2</sup> and Mossige,<sup>10</sup> together with section C12 of Guy's book,<sup>5</sup> provide a useful survey of results and bibliography on the problem up to 1981. Other recent work, all of which relates to the special case  $k = 3$ , includes that of Rødseth, Selmer and Rødne.<sup>13, 15, 16</sup>

We now modify the definition of  $(k, n, h)$ -PSS as follows. Choose positive integers  $k, h$  as before, and we call a set  $A = \{a_1, a_2, \dots, a_k\}$  a  $(k, n, h)$ -Modified Postage Stamp Set, or  $(k, n, h)$ -MPSS, iff, for every  $v \in \{1, 2, \dots, n\}$  there exists  $x_1, x_2, \dots, x_k$ , where each  $x_i$  is a (not necessarily positive) integer, such that:

$$(i) \quad v = \sum_{i=1}^k a_i x_i;$$

$$(ii) \quad \sum_{i=1}^k |x_i| \leq h.$$

Using our postage stamp analogy, we now allow any of our stamp denominations to occur a negative number of times, so long as the sum of the absolute values of the number of occurrences of each stamp type remains at most  $h$ .

In addition we extend other definitions in the obvious way so that, for given  $k$  and  $h$ , we call a  $(k, n, h)$ -MPSS having maximal  $n$  an *Extremal* MPSS, and this value of  $n$  we denote by  $n_h^*(k)$ . We also introduce some additional notation; if  $A$  is a set of  $k$  positive integers  $\{a_1, a_2, \dots, a_k\}$  with  $a_1 < a_2 < \dots < a_k$ , then let  $R_h(A)$  be the set:

$$\left\{ \sum_{i=1}^k a_i x_i : \sum_{i=1}^k |x_i| \leq h, x_i \text{ integers} \right\}$$

Then  $A$  is a  $(k, n, h)$ -MPSS iff  $\{1, 2, \dots, n\}$  is contained in  $R_h(A)$ .

The problem of constructing  $(k, n, 2)$ -MPSSs with maximal  $n$  can also be regarded as a natural analogue of Golomb's Ruler problem, a short introduction to which can be found in Dewdney's recent article,<sup>4</sup> and a survey can be found in Ref. 3. This problem has been well studied over the last few years because of its wide ranging applications in fields such as crystallography and the design of very large baseline radio interferometers. Note also that the version of this problem for the case where the differences  $a_j - a_i$  are computed modulo  $n$  relates strongly to the classical problem of finding *difference sets*; see for example chapter 2 of Hughes and Piper,<sup>8</sup> or chapter 11 of Hall.<sup>6</sup>

## 2. CONSTRUCTING MODIFIED POSTAGE STAMP SETS

We now consider some theoretical results concerning the existence of MPSSs. We are primarily concerned here with the case  $h = 2$ . We give some bounds on the value of  $n_h^*(k)$ , of which the lower bounds are obtained by explicit construction of corresponding MPSSs.

Even for the special case  $h = 2$ , the problem appears to be far from simple. Certainly the problem of computing the exact value of  $n_2^*(k)$  is a long way from being solved, even as far as asymptotic results are concerned, which clearly does not hold out much hope for obtaining explicit expressions for the value of  $n_h^*(k)$  for general  $h$  and  $k$ .

The fact that the problem does not appear easy is not really surprising, since the PSS problem is also a difficult one. As far as the value of  $n_2(k)$  is concerned, as long ago as 1937, Rohrbach<sup>14</sup> conjectured that  $n_2(k) = k^2/4 + O(k)$ , but this was subsequently disproved in 1976 by Hämerer and Hofmeister.<sup>7</sup> More recently Mossige,<sup>10</sup> and Mrose,<sup>12</sup> have shown that  $n_2(k) \geq 2k^2/7 + O(k)$ .

We first make some elementary observations, the proofs of which are straightforward.

**Lemma 2.1.** If  $A = \{a_1, a_2, \dots, a_k\}$  (with  $a_1 < a_2 < \dots < a_k$ ) is a  $(k, n, 2)$ -MPSS, then

- (i)  $A$  is a  $(k, m, 2)$ -MPSS for every  $m \leq n$ .
- (ii)  $n \leq \min\{2a_k, k^2 + k\}$ .
- (iii) If  $n > (k^2 - k)/2$  then  $n \geq a_1$ , and if  $n > (k^2 + k)/2$  then  $n \geq 2a_1$ .
- (iv) If  $n \geq a_1 - 1$  then  $a_k \geq 2a_1 - 1$ , and if  $n \geq 2a_1 - 1$  then  $a_k = 2a_1 - 1$  iff  $A = \{k, k + 1, \dots, 2k - 1\}$ .
- (v) If  $n \geq a_1 - 1$  then  $a_1 \leq (k^2 - k + 2)/2$ , and if  $n \geq 2a_1 - 1$  then  $a_1 \leq (k^2 + k + 2)/4$ .

From the above result we see that  $n_2^*(k)$  is bounded above by  $k^2 + k$ . In fact, if  $A$  is a  $(k, k^2 + k, 2)$ -MPSS then  $A$  is in some sense *perfect*, since every sum, difference and element of  $A$  is 'used' exactly once. However, it is easily seen that such perfect MPSSs cannot exist for  $k \geq 3$ , and so we actually have:

$$n_2^*(k) < k^2 + k.$$

The question that now naturally arises is: 'How big is  $n_2^*(k)$ ?'. The answer is by no means clear, although an unpublished construction of Hofmeister (1983) gives the lower bound

$$n_2^*(k) \geq k^2/2 + 0(k).$$

Hofmeister's construction is as follows.

Let  $h = 2, k \geq 2$ , and put  $x = [(k + 1)/2], y = 2x + 1$ . Then let  $A = \{1, 2, \dots, x, 3x + 1, 3x + 1 + 1 \cdot y, \dots, 3x + 1 + (k - x - 1) \cdot y\}$ .

It is then elementary to check that  $A$  is a  $(k, (k^2 + 3k)/2, 2)$ -MPSS for every  $k \geq 2$ , and hence Hofmeister's bound.

A slight modification of the above construction gives sets with slightly larger  $n$ . Let  $h, k, x$  be as above and put  $z = k - x + 1$ . Then let

$$A = \{z, 2z, \dots, x \cdot z, x \cdot z + 1, x \cdot z + 2, \dots, x \cdot z + z - 1\}$$

$A$  is then a  $(k, [k^2/2 + 2k], 2)$ -MPSS.

Using a slightly more complex construction method we now exhibit a lower bound for  $n_2^*(k)$  of  $k^2/2 + 3k + 2$  (for sufficiently large  $k$ ). Although a number of computer searches have failed to turn up any MPSSs with larger values of  $n$ , it would be premature to make any conjectures about the tightness of this bound, especially considering the history of Rohrbach's conjecture,<sup>14</sup> relating to a lower bound for  $n_2(k)$ .

Before proceeding to describe our method of construction, we need some preliminary definitions.

Suppose  $A = \{a_1, a_2, \dots, a_k\}$  (where  $a_1 < a_2 < \dots < a_k$ ) is a  $(k, n, 2)$ -MPSS, and in addition let  $t$  be the largest positive integer with the property that  $s \cdot a_1 \in A$  for every  $s \in \{1, 2, \dots, t\}$ . Then  $A$  is said to be an *Extensible*  $(k, n, 2)$ -MPSS iff:

- (i)  $n \geq \max\{2a_1, a_k\}$ , and
- (ii)  $\{a_1, a_1 + 1, a_1 + 2, \dots, a_k\}$  is contained in  $Q(A)$ , where  $Q(A)$  is defined to be the following subset of  $R_2(A)$ :

$$A \cup \{a_i + s \cdot a_1 : 1 \leq i \leq k, 1 \leq s \leq t\}$$

$$\cup \{a_i - s \cdot a_1 : 2 \leq i \leq k, 1 \leq s \leq t\}.$$

We can now state:

**Theorem 2.2.** Suppose that  $A = \{a_1, a_2, \dots, a_k\}$  is an extensible  $(k, n, 2)$ -MPSS with  $a_1 < a_2 < \dots < a_k$ . Then the set  $A^* = \{a_1\} \cup \{a_i + a_1 : 1 \leq i \leq k\}$  is an extensible  $(k + 1, n + 2a_1, 2)$ -MPSS.

*Proof.* We consider each value of  $v$  ( $1 \leq v \leq n + 2a_1$ ) in turn, and show that  $v \in R_2(A^*)$ . Moreover, if  $a_1 \leq v \leq a_k + a_1$ , then we simultaneously show that  $v \in Q(A^*)$ , and the result follows. We divide the proof into seven cases.

**Case 1:**  $1 \leq v < a_1$ . By definition of extensible we know that  $a_1 \leq n$ , and hence  $v \in R_2(A)$ . Also, since  $v < a_1$ , we know that  $v = a_j - a_i$  for some  $i, j$ . Hence  $v = (a_j + a_1) - (a_i + a_1) \in R_2(A^*)$ .

**Case 2:**  $v = a_1$ . By definition  $a_1 \in A^*$ , and hence  $a_1 \in Q(A^*)$ .

**Case 3:**  $a_1 < v < 2a_1$ . By definition of extensible we know that  $n \geq 2a_1$ , and hence  $v \in R_2(A)$ . By Lemma 2.1 (iv)  $v \leq a_k$ , and hence  $v \in Q(A)$ . Hence either: (a)  $v = a_i + s \cdot a_1$  ( $1 \leq s \leq t$ ), or (b)  $v = a_i - s \cdot a_1$  ( $1 \leq s \leq t$ ). If (a) then clearly  $v \in Q(A^*)$ , and if (b) then  $v = (a_i + a_1) - (s + 1) \cdot a_1 \in Q(A^*)$ .

**Case 4:**  $v = 2a_1$ . In this case  $v = a_1 + a_1 \in A^*$ , i.e.  $v \in Q(A^*)$ .

**Case 5:**  $2a_1 < v < a_1 + a_k$ . Let  $w = v - a_1$ , and then  $a_1 < w < a_k$ , and hence  $w \in Q(A)$ . Thus either: (a)  $w = a_i$  ( $1 \leq i \leq k$ ), (b)  $w = a_i + s \cdot a_1$  ( $1 \leq i \leq k, 1 \leq s \leq t$ ), or (c)  $w = a_i - s \cdot a_1$  ( $2 \leq i \leq k, 1 \leq s \leq t$ ). If (a) then  $v = a_i + a_1 \in A^*$ , i.e.  $v \in Q(A^*)$ . If (b) then  $v = a_i + (s + 1) \cdot a_1 \in Q(A^*)$ . If (c) then  $v = (a_i + a_1) - s \cdot a_1 \in Q(A^*)$ .

**Case 6:**  $v = a_1 + a_k$ . Immediately we have  $v \in A^*$ , i.e.  $v \in Q(A^*)$ .

**Case 7:**  $a_1 + a_k < v \leq n + 2a_1$ . Let  $w = v - 2a_1$ . Then  $a_k - a_1 < w \leq n$ . Now since  $w \leq n$  we know that  $w \in R_2(A)$ , and hence either: (a)  $w = a_i$  ( $1 \leq i \leq k$ ), or (b)  $w = a_i + a_j$  ( $1 \leq i, j \leq k$ ). If (a) then  $v = (a_i + a_1) + a_1 \in R_2(A^*)$ , and if (b) then  $v = (a_i + a_1) + (a_j + a_1) \in R_2(A^*)$ . ■

We call  $A^*$  the *extension* of  $A$ , and since  $A^*$  is itself extensible the whole process can be repeated any number of times. We use this technique to establish the existence of  $(k, n, 2)$ -MPSSs having  $n$  approximately equal to  $k^2/2 + 3k$ .

**Lemma 2.3.** If  $k$  is an even integer,  $k = 2r$  say, and  $k \geq 14$ , then

$$E(k) = \{k, k + 2\} \cup \{k + 5, k + 6, \dots, k + r - 1\} \cup \\ \{k + r + 1, k + r + 3\} \cup \{2k + 1, 2k + 3, 2k + 4\} \cup \\ \{2k + r, 2k + r + 2\} \cup \{2k + r + 4, 2k + r + 5, \dots, 3k - 1\}$$

is an extensible  $(k, 6k - 2, 2)$ -MPSS.

*Proof.* The proof follows from an exhaustive, and exhausting, case by case analysis, which there is no point in including here. ■

**Lemma 2.4.** If  $k$  is an odd integer,  $k = 2r + 1$  say, and  $k \geq 15$ , then

$$F(k) = \{k, k + 2\} \cup \{k + 5, k + 6, \dots, k + r - 1\} \cup \\ \{k + r + 1, k + r + 3, k + r + 4\} \cup \{2k + 1, 2k + 3, \\ 2k + 4\} \cup \\ \{2k + r, 2k + r + 2\} \cup \{2k + r + 5, 2k + r + 6, \dots, 3k - 1\}$$

is an extensible  $(k, 6k - 2, 2)$ -MPSS.

*Proof.* Immediate from a case by case analysis. ■

In addition to the above two lemmas the following are examples of extensible  $(k, 6k - 2, 2)$ -MPSSs with  $k = a_1$  for  $k = 10, 12$  and  $13$ :

$$k = 10: \{10, 12, 15, 16, 21, 23, 24, 27, 28, 29\}$$

$$k = 12: \{12, 13, 14, 18, 19, 27, 28, 29, 32, 33, 34, 35\}$$

$$k = 13: \{13, 14, 15, 19, 20, 22, 29, 30, 31, 34, 36, 37, 38\}.$$

The following are examples of extensible  $(k, n, 2)$ -MPSSs with maximal  $n$  (given  $k = a_1$ ) for the remaining values of  $k$ :

- $k = 2 (n = 6): \{2, 3\}$
- $k = 3 (n = 10): \{3, 4, 5\}$
- $k = 4 (n = 16): \{4, 6, 7, 9\}$
- $k = 5 (n = 20): \{5, 7, 8, 9, 11\}$
- $k = 6 (n = 28): \{6, 9, 10, 11, 13, 14\}$
- $k = 7 (n = 32): \{7, 10, 11, 12, 13, 15, 16\}$
- $k = 8 (n = 42): \{8, 12, 14, 17, 18, 19, 21, 23\}$
- $k = 9 (n = 50): \{9, 10, 14, 17, 20, 21, 22, 24, 25\}$
- $k = 11 (n = 62): \{11, 12, 15, 17, 21, 24, 25, 27, 29, 30, 31\}$ .

We now have:

*Theorem 2.5.* If  $k = 16, 17, 18$  or  $k \geq 20$ , then:

$$n_2^*(k) \geq k^2/2 + 3k + 2.$$

*Proof.* Suppose that  $A$  is an extensible  $(k, 6k - 2, 2)$ -MPSS having minimum element  $k$ . Then, if  $A$  is extended  $t$  times (as in Theorem 2.2), we obtain an extensible  $(k+t, 6k-2+tk, 2)$ -MPSS. If we let  $K = k+t$ , then:

- if  $t = k - 4$  or  $k - 2$  we obtain a  $(K, K^2/2 + 3K + 2, 2)$ -MPSS, and
- if  $t = k - 3$  we obtain a  $(K, K^2/2 + 3K + 5/2, 2)$ -MPSS.

Using Lemmas 2.3 and 2.4 and the list following these lemmas the result follows. ■

The following result gives the best lower bound known to the author for all other values of  $k$ .

*Theorem 2.6.* For  $2 \leq k \leq 15$  and  $k = 19, n_2^*(k)$  is bounded below by the values in Table 1. For  $k \leq 8$  these appear to be the actual values of  $n_2^*(k)$ .

*Proof.* MPSSs with the appropriate parameters can be constructed from the examples given immediately above Theorem 2.5 (hence establishing the lower bounds).

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Table 1. Values of  $n_2^*(k)$

$k$	bound	$k$	bound	$k$	bound
2	6	7	40	12	106
3	10	8	52	13	122
4	16	9	64	14	140
5	24	10	76	15	158
6	32	11	90	19	238

The fact that these bounds appear tight for  $k \leq 8$  follows from computer searches. ■

It is interesting to note that for every value of  $k$ , at least one of the MPSSs having largest known  $n$  is extensible. This may be an artefact of the scale of the searches that have been undertaken; however, it is an interesting subject for further research. Values of  $k$  do exist, however, for which non-extensible MPSSs are known to exist having  $n$  as large as any known extensible MPSSs; for example:

- $\{1, 5, 8\}$  is a  $(3, 10, 2)$ -MPSS,
- $\{1, 2, 8, 13\}$  is a  $(4, 16, 2)$ -MPSS,
- $\{5, 7, 12, 13, 15, 16\}$  is a  $(6, 32, 2)$ -MPSS, and
- $\{2, 11, 13, 14, 18, 19, 21\}$ ,  $\{2, 13, 14, 16, 17, 22, 23\}$ ,  $\{5, 11, 13, 17, 19, 20, 40\}$  and  $\{7, 9, 15, 17, 19, 20, 40\}$  are all  $(7, 40, 2)$ -MPSSs.

Certainly, some further directed computer searches may resolve some of the obvious questions arising from this work. It also remains interesting to see how many theoretical results can be achieved in an area in which they have always been thin on the ground.

Acknowledgements

The author would like to express his thanks to an anonymous referee for a number of valuable suggestions for improvement, including reference to Hofmeister's earlier work. The author would also like to point out that this paper would have been much the poorer without the assistance of Greg Watson and Alex Selby, who wrote a number of programs to perform searches of various types during the research described in this paper.

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