One-Stage One-Sided Rearrangeable Switching Networks

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Abstract—**We consider switching networks consisting of subscriber lines and crosswires connected by switches. A connection between two subscribers is made along one crosswire via two switches. We determine the minimum number of switches necessary for such a switching network to be rearrangeably nonblocking and construct a switching arrangement which achieves this minimum for any (even) number of subscriber lines. Two procedures for assignment of crosswires to subscriber line pairs are described. One makes correct choice of connection route without backtracking provided all connections are known beforehand; the other determines a rearrangement of existing assignments when a new connection is required. We characterize the switching networks which have the minimum number of switches for networks with up to eight subscriber lines and give nonisomorphic solutions for larger networks.**

I. Introduction

In two recent papers, [1], [2], Newbury and Raby have considered the use of one-stage switching arrangements in bidirectional telephone systems. When the number of subscriber lines is small such arrangements have the potential to provide networks requiring the least number of switches. A switching network for eight subscriber lines using the minimum number of switches is described in [2]. In this kind of network, each subscriber line is connected via switches to some (or all) of a set of crosswires. Thus, a connection between two subscribers is made along one crosswire via two switches. If there are $2n$ subscribers then n crosswires are required to connect them simultaneously in n pairs. We refer to an arrangement of $2n$ subscriber lines, n crosswires and the switches connecting them as a one-stage one-sided switching network and denote it SN(n). If the $2n$ subscribers may be simultaneously connected in pairs for all possible partitions of them into n pairs then the switching network is said to be rearrangeably nonblocking (or rearrangeable) and is denoted RSN(n). A SN(n) is rearrangeable if for every partition of the subscribers into n pairs there is a bijective assignment of the crosswires to the pairs such that each subscriber line is connected to the crosswire assigned to it.

An incidence structure (Hughes and Piper [3]) is a triple (P, B, S) where P and B are disjoint finite sets of objects (called points and blocks, respectively) and $S \subseteq P \times B$ is an incidence relation between them. If $(P, x) \in S$, we say P is incident with x, P is on x or x contains P. A switching network may be described by an incidence structure. Let the subscribers be labeled with $P = \{P_1, P_2, \cdots, P_{2n}\}$ and the crosswires with $B = \{x_1, x_2, \cdots, x_n\}$, and let S be the set of pairs $(P_i, x_j) \in P \times B$ such that subscriber P_i is connected to crosswire x_j by a switch in a given SN(n). Then $D = (P, B, S)$ is an incidence structure which completely describes the switching network and we identify the two. The incidence structure D may be represented by a $2n$ by n incidence matrix $A = (a_{ij})$ where $a_{ij} = 1$ if $(P_i, x_j) \in S$ and $a_{ij} = 0$ otherwise. Any partition of P into n pairs may be written as $\{P_{f(1)}, P_{f(2)}\}$ $\{P_{f(3)}, P_{f(4)}\}, \cdots, \{P_{f(2n-1)}, P_{f(2n)}\}$ for some permutation f of $\{1, 2, \cdots, 2n\}$. The switching network is rearrangeable if for any such partition there is a permutation g of $\{1, 2, \cdots, n\}$ such that $(P_{f(2i-1)}, x_{g(i)})$ and $(P_{f(2i)}, x_{g(i)})$ belong to S for $i = 1, 2, \cdots, n$. Thus, the crosswire $x_{g(i)}$ is assigned to the pair $\{P_{f(2i-1)}, P_{f(2i)}\}$ and subscribers $P_{f(2i-1)}$ and $P_{f(2i)}$ are connected via crosswire $x_{g(i)}$.

In this paper, we consider the minimum number of switches required in an RSN(n). We show that if (P, B, S) is an RSN(n) then $|S| \geq n^2 + 2n - 1$. Moreover, for every $n \geq 1$, we construct an RSN(n) with the minimum number $n^2 + 2n - 1$ of switches. When $n > 2$ this number is smaller than the number of switches in a simple triangular arrangement (see [1]). We describe a procedure for making the assignment of to subscriber pairs and find all RSN(n) with the minimum number of switches for $n \leq 4$.

II. Systems of Distinct Representatives

In this section, we use Hall's Marriage Theorem to give necessary and sufficient conditions that a SN(n) be rearrangeable.

Let $D = (P, B, S)$ be a SN(n). Let Q be any subset of P. We write (Q) for the set of crosswires which are connected to each of the members of Q by a switch. $(Q) = \{x_j \in B | (P_i, x_j) \in S \text{ for all } P_i \in Q\}$. Similarly if $Y \subseteq B$ then $(Y) = \{P_i \in P | (P_i, x_j) \in S \text{ for all } x_j \in Y\}$. When $Q = \{P_i\}$ is a singleton, we write (P_i) instead of $(\{P_i\})$. Similarly, we write (x_j) for $(\{x_j\})$. Let $r_i = |(P_i)|$ denote the number of elements in (P_i) and let $k_j = |(x_j)|$ denote the number of elements in (x_j). Any partition $Q_1 = \{P_{f(1)}, P_{f(2)}\}, \cdots, Q_n = \{P_{f(2n-1)}, P_{f(2n)}\}$ of P into n pairs determines n subsets $Y_1 = (Q_1), \cdots, Y_n = (Q_n)$ of B. Now D is rearrangeable if for every such partition of P there is a system of distinct representatives from the n sets $Y_1, \cdots, Y_n$; that is, there are distinct elements $y_1, \cdots, y_n$ of B such that $y_i \in Y_i$ for $i = 1, \cdots, n$. Then $y_1, \cdots, y_n$ are necessarily all the elements of B and so $y_i = x_{g(i)}$ for some permutation g of $\{1, 2, \cdots, n\}$. Hall [4] has given necessary and sufficient conditions that subsets $A_1, A_2, \cdots, A_s$ of a set A have a system of distinct representatives.

Theorem 1: (Halls Marriage Theorem). Let $A_1, \cdots, A_s$ be subsets of a set A. Then there exist distinct elements $a_1, a_2, \cdots, a_s \in A$ such that $a_1 \in A_1, a_2 \in A_2, \cdots, a_s \in A_s$ if and only if every t, $1 \leq t \leq s$, and every t subset

$$\{i_1, \cdots, i_t\} \subseteq \{1, 2, \cdots, s\},$$

$$\cdot\ |A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_t}| \geq t.$$

Let $D = (P, B, S)$ be a SN(n). We say D has property $T(s)$ if for any s disjoint two subsets $Q_1 = \{P_{i_1}, P_{i_2}\}, \cdots, Q_s = \{P_{i_{2s-1}}, P_{i_{2s}}\}$ of P we have $|(Q_1) \cup \cdots \cup (Q_s)| \geq s$.

Proposition 1: Let $D = (P, B, S)$ be a SN(n). Then D is rearrangeable if and only if D has property $T(s)$ for $s = 1, 2, \cdots, n$.

Proof: Suppose D is rearrangeable. Let $Q_1, \cdots, Q_s$ be disjoint two subsets of P. Consider any partition of P into n

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two subsets which contain the s two subsets $Q_1, \cdots, Q_s$. As D is rearrangeable there is an assignment of the crosswires to the two subsets of this partition. Thus, there are distinct elements $x_{j_1}, \cdots, x_{j_s}$ of B such that $x_{j_l} \in (Q_l)$ for $l = 1, 2, \cdots, s$. Hence, $|(Q_1) \cup \cdots \cup (Q_s)| \geq s$ and D has property $T(s)$. This holds for $s = 1, 2, \cdots, n$. Conversely, suppose D has property $T(s)$ for $s = 1, 2, \cdots, n$. Let $Q_1, \cdots, Q_n$ be a partition of P into two subsets. The $(Q_1), \cdots, (Q_n)$ are subsets of B such that for any $s(1 \leq s \leq n)$, $|(Q_{i_1}) \cup \cdots \cup (Q_{i_s})| \geq s$ for all $\{i_1, \cdots, i_s\} \subseteq \{1, 2, \cdots, n\}$. Hence, by Hall's Marriage Theorem there exists a system of distinct representatives from the sets $(Q_1), \cdots, (Q_n)$. This determines an assignment of the crosswires to the two subsets $Q_1, \cdots, Q_n$ and hence D is rearrangeable.

The augmenting matching algorithm (see, for example, Brualdi [5], Tucker [6]) may be used to find the system of distinct representatives when it exists. Given an RSN(n) and a partition $Q_1 = \{P_{f(1)}, P_{f(2)}\}, \cdots, Q_n = \{P_{f(2n-1)}, P_{f(2n)}\}$ of P into n subsets this algorithm finds a permutation g such that $x_{g(i)} \in (Q_i)$. Furthermore, given any partial assignment of crosswires to subscriber pairs it will determine new assignments when further connections between subscribers are required. Indeed, suppose we have s disjoint two subsets $Q_1 = \{P_{f(1)}, P_{f(2)}\}, \cdots, Q_s = \{P_{f(2s-1)}, P_{f(2s)}\}$ of P and a partial assignment of $x_{g(1)}, \cdots, x_{g(s)}$, respectively, to these subsets where $x_{g(i)} \in (Q_i)$ for $i = 1, \cdots, s$. Let $Q_{s+1} = \{P_{f(2s+1)}, P_{f(2s+2)}\}$ be any two subset of P disjoint from $Q_1, \cdots, Q_s$. Then the algorithm finds a sequence $i_o, i_1, \cdots, i_s = s + 1$ such that Q_{i_0} contains a crosswire, denoted $x_{h(i_o)}$, not among $x_{g(1)}, \cdots, x_{g(s)}$ and $x_{g(i_j)} \in (Q_{i_{j+1}})$ for $j = 0, 1, \cdots, s - 1$. Put $h(i_{j+1}) = g(i_j)$ for $j = 0, \cdots, s - 1$ and $h(l) = g(l)$ for $l \in (1, \cdots, s) \setminus \{i_o, \cdots, i_s\}$.

Then h determines a partial assignment for the $s + 1$ two subsets $Q_1, \cdots, Q_{s+1}$. Thus, by making a connection along $x_{h(i_j)}$ and breaking the connection along $x_{g(i_j)}$ between $P_{f(2i_j-1)}$ and $P_{f(2i_j)}$ successively for $j = 0, \cdots, s - 1$ one may then make a connection between $P_{f(2s+1)}$ and $P_{f(2s+2)}$ along $x_{h(i_s)}$ so that the $s + 1$ pairs $Q_1, \cdots, Q_{s+1}$ may all be connected without any interruption to the existing connections. (Of course when $s = 0$, no rearrangement of connections is required and subscribers $P_{f(2s+1)}$ and $P_{f(2s+2)}$ may be connected by $x_{h(i_o)}$ directly.)

III. Minimum Number of Switches

We obtain a bound on the number $|S|$ of switches in an RSN(n) by means of two lemmas which give necessary and sufficient conditions that a SN(n) has property $T(n - 1)$ and property $T(n)$.

Remark 1: Let $D = (P, B, S)$ be an incidence structure and suppose $P \in P$ and $x \in B$. Then $P \in (x)$ if and only if $x \in (P)$. This is because both these statements hold if and only if $(P, x) \in S$.

Lemma 1: Let $D = (P, B, S)$ be a SN(n). Then property $T(n)$ holds in D if and only if $k_j \geq n + 1$ for all $x_j \in B$ where $k_j = |(x_j)|$ is the number of elements of (x_j).

Proof: Suppose $k_j \geq n + 1$ for all $x_j \in B$. Let $Q_1, \cdots, Q_n$ be a partition of P into n two subsets. Let $x_j \in B$. Since (x_j) contains at least $n + 1$ elements and there are only n subsets in the partition, at least one of the two subsets, Q_i say, is contained in (x_j). But then for each $x_j \in B$ we have $x_j \in (Q_i)$ for some i and so $B \subseteq (Q_1) \cup \cdots \cup (Q_n)$. Hence, property $T(n)$ holds.

Conversely, suppose $k_j = s \leq n$ for some x_j. Let $(x_j) = \{P_{i_1}, \cdots, P_{i_s}\}$ and consider any partition $Q_1 = \{P_{f(1)}, P_{f(2)}\}, \cdots, Q_n = \{P_{f(2n-1)}, P_{f(2n)}\}$ where $f(2j) = i_j$ for $j = 1, \cdots, s$. Then $x_j \notin (Q_i)$ for $i = 1, \cdots, n$ and so $|(Q_1) \cup \cdots \cup (Q_n)| \leq n - 1$. Thus, property $T(n)$ does not hold.

Lemma 2: Let $D = (P, B, S)$ be a SN(n). Then property $T(n - 1)$ holds in D if and only if there is at most one $x_j \in B$ such that $k_j < n + 2$.

Proof: Suppose that there is at most one $x_j \in B$ such that $k_j < n + 2$. Let $Q_1, \cdots, Q_{n-1}$ be $n - 1$ disjoint two subsets of P. If $k_j \geq n + 2$ then (x_j) contains at least n elements belonging to the union of the $n - 1$ disjoint two subsets $Q_1, \cdots, Q_{n-1}$.

Hence, $Q_i \subseteq (x_j)$ for some $i(1 \leq i \leq n - 1)$ and so $x_j \in (Q_i)$. Since $k_j \geq n + 2$ for at least $n - 1$ elements $x_j \in B$ we have $|(Q_1 \cup \cdots \cup (Q_{n-1})| \geq n - 1$. Thus, property $T(n - 1)$ holds. Conversely, suppose that there are two elements x_{j_1} and x_{j_2} of B such that $k_{j_1} = s_1 \leq n + 1$ and $k_{j_2} = s_2 \leq n + 1$.

Without loss of generality we may assume that $s_1 \geq s_2$. Let $|(x_{j_1}) \cap (x_{j_2})| = s$ and put $(x_{j_1}) \setminus (x_{j_2}) = \{P_{a_1}, \cdots, P_{a_{s-s_1}}\}$,

$$(x_{j_1}) \cap (x_{j_2}) = \{P_{a_{s_1-s+1}}, \cdots, P_{a_{s_1}}\}$$

and

$$(x_{j_2})/(x_{j_1}) = \{P_{b_1}, \cdots, P_{b_{s_2-s}}\}.$$

Let f be any permutation of $\{1, 2, \cdots, 2n\}$ such that $f(i) = a_i$ for $i = 1, \cdots, s$, and $f(2n + 1 - i) = b_i$ for $i = 1, \cdots, s_2 - s$ and consider the partition $Q_1 = \{P_{f(1)}, P_{f(2n)}\}$, $Q_2 = \{P_{f(2)}, P_{f(2n-1)}\}, \cdots, Q_n = \{P_{f(n)}, P_{f(n+1)}\}$. Then $x_{j_1}, x_{j_2} \notin (Q_i)$ for $i = 1, 2, \cdots, n - 1$.

Hence, $|(Q_1) \cup \cdots \cup (Q_{n-1})| \leq n - 2$ and property $T(n - 1)$ does not hold.

Theorem 2: Let $D = (P, B, S)$ be an RSN(n). Then $|S| \geq n^2 + 2n - 1$. If $|S| = n^2 + 2n - 1$ then there is a unique $x_l \in B$ with $k_l = n + 1$ and $k_j = n + 2$ for all $x_j \in B \setminus \{x_l\}$.

Proof: As D is an RSN(n), properties $T(n - 1)$ and $T(n)$ hold in D. By Lemma 1, $k_j \geq n + 1$ for all $x_j \in B$ and by Lemma 2, 2, $k_j \geq n + 2$ for at least $n - 1$ crosswires $x_j \in B$. Hence,

$$|S| = \sum_{j=1}^{n} k_j \geq n + 1 + (n-1)(n+2) = n^2 + 2n - 1.$$

If equality holds then $k_j = n + 2$ for exactly $n - 1$ crosswires $x_j \in B$ and $k_l = n + 1$ for the remaining crosswire x_l.

An RSN(n) with the minimum number $n^2 + 2n - 1$ of switches is called minimal. A minimal RSN(n) has $n + 1$ switches on one crosswire and $n + 2$ switches on each of the remaining $n - 1$ crosswires.

IV. Construction of Minimal RSN(n)

We first consider the case of even n. Let $n = 2m$ be even and let $M_1(2m)$ denote the SN(n), (P, B, S_1) where the elements belonging to S_1 are as follows. For $1 \leq i \leq m$, $(P_{2i-1}, x_i) \in S_1$, $(P_{2i}, x_i) \in S_1$, and $(P_j, x_i) \in S_1$ for $2m + 1 \leq j \leq 4m$; for $m + 1 \leq i \leq 2m$, $(P_{2i-1}, x_i) \in S_1$, $(P_{2i}, x_i) \in S_1$ and $(P_j, x_i) \in S_1$ for $1 \leq j \leq 2m$ with the exception that $(P_{4m}, x_{2m}) \notin S_1$.

Below is the incidence matrix of $M_1(6)$.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

We claim that $M_1(2m)$ is a minimal RSN$(2m)$. Clearly, $|S_1| = (2m-1)(2m+2) + 2m + 1 = 4m^2 + 4m - 1 = n^2 + 2n - 1$. We show that $M_1(2m)$ is rearrangeable by showing that property $T(s)$ holds for $s = 1, 2, \cdots, n$. Let $1 \le s \le n$ and let $Q_1 = \{P_{f(1)}, P_{f(2)}\}, \cdots, Q_s = \{P_{f(2s-1)}, P_{f(2s)}\}$ be any s disjoint two subsets of P. We show that $|(Q_1) \cup \cdots \cup (Q_s)| \ge s$.

If $Q = \{P_{i_1}, P_{i_2}\}$ with $i_1 < i_2$ is a two subset of P we say Q is of type (1), (2), (3), or (4) according as

1) $1 \le i_1 < i_2 \le 2m$; (e.g., rows 1 and 3 of A above),

2) $2m + 1 \le i_1 < i_2 \le 4m$; (e.g., rows 7 and 9 of A above),

3) $1 \le i_1 \le 2m < i_2 \le 4m - 1$; (e.g., rows 1 and 7 of A above), and

4) $1 \le i_1 \le 2m$ and $i_2 = 4m$; (e.g., rows 1 and 12 of A above).

We use this notion of type throughout whenever describing the two subsets of P in $M_1(2m)$.

Remark 2: If Q is of type 1) then $(Q) \supseteq \{x_{m+1}, \cdots, x_{2m}\}$. If Q is of type 2) then $(Q) \supseteq \{x_1, \cdots, x_m\}$. If Q is of type 3) then $(Q) = \{x_{j_1}, x_{j_2}\}$ where $i_1 = 2j_1 - 1$ or $2j_1$ and $i_2 = 2j_2 - 1$ or $2j_2$. If Q is of type 4) then $(Q) = \{x_j\}$ where $i_1 = 2j - 1$ or $2j$.

We consider four cases.

Case 1: Among $Q_1, \cdots, Q_s$ there is at least one two subset of type 1) and at least one two subset of type 2). Then $(Q_1) \cup \cdots \cup (Q_s) = B$. Hence, in this case $|(Q_1) \cup \cdots \cup (Q_s)| = 2m \ge s$ and property $T(s)$ holds.

Case 2: Among $Q_1, \cdots, Q_s$ there are $l \ge 1$ two subsets, $Q_1, \cdots, Q_l$, say of type 1) and the remaining $s - l$ two subsets are of type 3) or 4). Then, $Q_1 \cup \cdots \cup Q_s$ contains $2l + s - l = s + l$ elements of $\{P_1, \cdots, P_{2m}\}$. Hence, $s + l \le 2m$. Now $(Q_i) \supseteq \{x_{m+1}, \cdots, x_{2m}\}$ for $i = 1, \cdots, l$. Any x_j with $1 \le j \le m$ belongs to at most two of $(Q_{l+1}), \cdots, (Q_s)$ and each of $(Q_{l+1}), \cdots, (Q_s)$ contains exactly one x_j with $1 \le j \le m$.

Thus, $|(Q_1) \cup \cdots \cup (Q_s)| \ge m + (s-l)/2 \ge (s+l)/2 + (s-l)/2 = s$. Hence, in this case also property $T(s)$ holds.

Case 3: Among $Q_1, \cdots, Q_s$ there are $l \ge 1$ two subsets, $Q_1, \cdots, Q_l$ say, of type 2) and the remaining $s - l$ two subsets are of type 3) or 4). Then, $Q_1 \cup \cdots \cup Q_s$ contains $2l + s - l = s + l$ elements of $\{P_{2m+1}, \cdots, P_{4m}\}$. Hence, $s + l \le 2m$. Now $(Q_i) \supseteq \{x_1, \cdots, x_m\}$ for $i = 1, \cdots, l$. Any x_j with $m + 1 \le j \le 2m$ belongs to at most two of $(Q_{l+1}), \cdots, (Q_s)$ and with possibly one exception [when one of $Q_{l+1}, \cdots, Q_s$ is of type (4)] each of $(Q_{l+1}), \cdots, (Q_s)$ contains exactly one x_j with $m + 1 \le j \le 2m$.

Thus, $|(Q_1) \cup \cdots \cup (Q_s)| \ge m + (s-l-1)/2 \ge (s+l)/2 + (s-l-1)/2 = s - 1/2$. Hence, in this case, also $|(Q_1) \cup \cdots \cup (Q_s)| \ge s$ and property $T(s)$ holds.

Case 4: None of $Q_1, \cdots, Q_s$ are of type 1) or 2). Then each of $x_1, \cdots, x_{2m}$ belongs to at most two of $(Q_1), \cdots, (Q_s)$. Now (Q_i) contains two elements of B if Q_i is of type 3) and one such element if Q_i is of type 4). At most one of $Q_1, \cdots, Q_s$ is of type 4) so that

$$|(Q_1) \cup \cdots \cup (Q_s)| \ge \frac{2(s-1)+1}{2} = s - \frac{1}{2}.$$

Hence, in this case also $|(Q_1) \cup \cdots \cup (Q_s)| \ge s$ and property $T(s)$ holds.

We have shown that property $T(s)$ holds in all cases. Hence, property $T(s)$ holds in $M_1(2m)$ for $s = 1, 2, \cdots, 2m$ and $M_1(2m)$ is a minimal RSN$(2m)$.

We now show that there is substructure of $M_1(2m)$ which is a minimal RSN$(2m-1)$ and hence we will have constructed a minimal RSN(n) for all $n \ge 1$ both even and odd.

Let $D = (P, B, S)$ be an RSN(n). Let $P_i, P_j \in P$ and $x_l \in B$ with $x_l \in (\{P_i, P_j\})$ and consider the substructure (P', B', S') where $P' = P \setminus \{P_i, P_j\}$, $B' = B \setminus \{x_l\}$ and $S' = \{(P_s, x_t) \in S \mid P_s \in P', x_t \in B'\}$. Then (P', B', S') is a SN$(n-1)$ and we denote it $D|\{P_i, P_j, x_l\}$. Since every partition of P into $n - 1$ two subsets corresponds to a partition of P into n two subsets one of which is $\{P_i, P_j\}$, $D|\{P_i, P_j, x_l\}$ is rearrangeable if and only if for any partition of P into two subsets one of which is $\{P_i, P_j\}$ there is an assignment of the crosswires to the two subsets which assigns x_l to $\{P_i, P_j\}$.

Proposition 2: Let $D = (P, B, S)$ be an R2SBN(n). Let $P_i, P_j \in P$ and suppose $(\{P_i, P_j\}) = \{x_l\}$. Then $D|\{P_i, P_j, x_l\}$ is an RSN$(n-1)$.

Proof: This follows immediately since, as $(\{P_i, P_j\}) = \{x_l\}$, for any partition of P into n two subsets one of which is $\{P_i, P_j\}$ any assignment of crosswires to two subsets must assign x_l to $\{P_i, P_j\}$

We note that in $M_1(2m)$, $(\{P_{2m}, P_{4m}\}) = \{x_m\}$. It follows that $M(2m-1) = (P', B', S') = M_1(2m)|\{P_{2m}, P_{4m}, x_m\}$ is an RSN$(2m-1)$. Moreover, $M(2m-1)$ is minimal. For S' is obtained from S_1 by removing (P_{2m}, x_j), $m + 1 \le j \le 2m$; (P_{4m}, x_j), $1 \le j \le m - 1$; and (P_i, x_m), $2m - 1 \le i \le 4m$. Thus, $|S'| = 4m^2 + 4m - 1 - (m + m - 1 + 2m + 2) = 4m^2 - 2 = (2m-1)^2 + 2(2m-1) - 1$.

V. Assignment Procedure

We have seen that the augmenting matching algorithm may be used to make an assignment of crosswires to the two subsets of a partition of P in an RSN(n). As successive assignments are made this may involve some rearrangement of connections. Newbury [2] has given an assignment procedure for $M_1(4)$ which allows correct choice of connection when the four two subsets of a partition are known beforehand. We describe a procedure which allows correct connection in $M_1(2m)$ given the $2m$ two subsets of a partition. At each step an assignment of a crosswire is made to an unassigned two subset until all $2m$ two subsets have crosswire assigned. At no stage is there any need for backtracking to make a different assignment in order to complete all $2m$ assignments. This procedure may be adapted to give a procedure for $M(2m-1)$.

We consider $M_1(2m) = (P, B, S)$. If Q is a two subset of P and $B_1 \subseteq \{x_1, \cdots, x_m\}$ and $B_2 \subseteq \{x_{m+1}, \cdots, x_{2m}\}$ are given subsets of B then the p value of Q is the number of elements of (Q) in $B_1 \cup B_2$. As described above Q will be of type 1), 2), 3), or 4) depending on which two elements of P belong to Q.

Let $Q_1, \cdots, Q_{2m}$ be a partition of P into $2m$ two subsets. We begin our assignment procedure by making a list of the two subsets of type 3) or 4) among $Q_1, \cdots, Q_{2m}$. Since the number of two subsets of type 1) among $Q_1, \cdots, Q_{2m}$ equals the number of two subsets of type 2), we begin with an even number of two subsets in our list. We set $B_1 = \{x_1, \cdots, x_m\}$ and $B_2 = \{x_{m+1}, \cdots, x_{2m}\}$ and assign crosswires to the two subsets in the list by the following procedure.

While list not empty do
  begin
  if some Q_i in the list has p value equal to 1 then
    assign x_j to Q_i where $(Q_i) = \{x_j\}$
  else
    for the first two subset Q_i in the list assign x_j to Q_i where $x_j \in (Q_i)$ and x_j belongs to the larger of the sets B_1 and B_2 or either set if B_1 and B_2 have equally many members
  remove Q_i from the list and x_j from the appropriate set B_1 or B_2
  end.

We begin this procedure with all p values of two subsets in the list equal to 2 with possibly one exception (with p value equal to 1) and since each crosswire belongs to at most two two subsets of type 3) or 4) among $Q_1, \cdots, Q_{2m}$ this remains true throughout. Thus, the procudure continues until a crosswire is assigned to each two subset of type 3) or 4) among $Q_1, \cdots, Q_{2m}$. Now (Q_i) contains one crosswire from $\{x_1, \cdots, x_m\}$ and one from $\{x_{m+1}, \cdots, x_{2m}\}$ for each two subset Q_i of type (3). Hence, except possibly on the first occasion, when an assignment of a crosswire is made to a two subset with p value 1, that crosswire and the crosswire assigned on the preceding assignment come one from B_1 and one from B_2. Hence, as we begin with an even number of two subsets in the list, the procedure terminates with equally many crosswires remaining in B_1 as in B_2. We complete the assignment procedure by arbitrarily assigning the crosswires remaining in B_2 to the two subsets of type 1) among. $Q_1, \cdots, Q_{2m}$ and the crosswires remaining in B_1 to the two subsets of type 2) among $Q_1, \cdots, Q_{2m}$. This works because, as we observed above, the number of two subsets of type 1) among $Q_1, \cdots, Q_{2m}$ equals the number of type 2).

VI. Nonisomorphic Minimal RSN(n)

Two SN(n), $D_1 = (P, B, S_1)$ and $D_2 = (P, B, S_2)$ are isomorphic if there are permutations α of P and β of B such that $(P_i, x_j) \in S_1$ if and only if $(\alpha(P_i), \beta(x_j)) \in S_2$. Thus, D_1 and D_2 are isomorphic when the corresponding switching arrangements are the same up to reordering (or relabeling) of the subscriber lines and the crosswires. We have constructed a minimal RSN(n) for every $n \geq 1$. There may, however, for any given n, be many nonisomorphic minimal RSN(n). We show that there are exactly three distinct RSN(4) and describe a minimal RSN($2m$) which is not isomorphic to $M_1(2m)$ for $m \geq 3$.

Consider $M_1(2m) = (P, B, S_1)$. Putting $S = S_1 \cup \{P_{4m}, x_{2m})\}$ we see that (P, B, S) is an RSN($2m$) with a great deal of symmetry. In fact, removing any switch of the form (P_i, x_j) where $m + 1 \leq j \leq 2m$ if $1 \leq i \leq 2m$ and $1 \leq j \leq m$ if $2m + 1 \leq i \leq 4m$ we obtain a SN(n) isomorphic to $M_2(2m) = (P, B, S_2)$ where $S_2 = S \setminus \{(P_{4m}, x_m)\}$. Similar arguments to those given above for $M_1(2m)$ show that $M_2(2m)$ is a minimal RSN($2m$). Moreover, for $m \geq 3$ $M_2(2m)$ is not isomorphic to $M_1(2m)$ which is itself isomorphic to any RSN(n) obtained by removing from S any switch of the form (P_i, x_j) where $i = 2j - 1$ or 2j.

We say that a SN(n) is in standard form if $r_1 \leq r_2 \leq \cdots \leq r_{2n}$ and $k_1 \geq k_2 \geq \cdots \geq k_n$. Clearly, every SN($n$) is isomorphic to a SN($n$) in standard form. We show that when $n = 1$, 2, or 3 any two minimal RSN(n) in standard form are isomorphic. Hence, up to isomorphism there is just one minimal RSN(n) for $n = 1$, 2, or 3. Let $D = (P, B, S)$ be a minimal RSN(n) in standard form. By Theorem 2 $k_i = n + 2$ for $i = 1, 2, \cdots, n - 1$ and $k_n = n + 1$. When $n = 1$ D has incidence matrix

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and is isomorphic to $M(1)$. When $n = 2$ D has incidence matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and is isomorphic to $M_1(2)$ [and $M_2(2)$].

The following lemma shows that these are the only two cases where $r_i = 1$ for some $P_i \in P$.

Lemma 3: Let $D = (P, B, S)$ be a minimal RSN(n) with $n \geq 3$. Then $r_i \geq 2$ for all $P_i \in P$.

Proof: Let $P_i \in P$. As property $T(1)$ holds we have $(\{P_i, P_j\} \neq \Phi$ for all $P_j \in P \setminus \{P_i\}$. Hence, $(P_i) \neq \Phi$. Let $x_l \in (P_i)$. Now $k_l \leq n + 2 < 2n$. Hence, there is a $P_j \in P$ such that $P_j \notin (x_l)$. Further, as $(\{P_i, P_j\}) \neq \Phi$ and $x_l \notin (\{P_i, P_j\})$ there is an $x_m \in (\{P_i, P_j\}) \subseteq (P_i)$ with $x_m \neq x_l$. Thus, $\{x_l, x_m\} \subseteq (P_i)$ and $r_i \geq 2$.

Now when $n = 3$, since $|S| = 14$ the minimal RSN(n) in standard form D has $r_i = 2$ for $1 \leq i \leq 4$ and $r_i = 3$ for $5 \leq i \leq 6$. It is then straightforward to see that D must be isomorphic to an RSN(3) with incidence matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and D is isomorphic to $M(3)$.

The following proposition characterizes minimal RSN(n) with $n \geq 4$ and $r_i = 2$ for more than one $P_i \in P$.

Proposition 3: Let $D = (P, B, S)$ be a minimal RSN(n), $n \geq 4$, in standard form. Suppose $r_1 = r_2 = 2$. Then $n = 4$, $r_3 = 2$ and $\{P_1, P_2, P_3\} \subseteq (x_4)$.

Proof: As property $T(1)$ holds in D, $(P_1) \cap (P_2) \neq \Phi$. Suppose $(P_1) = (P_2) = \{x_p, x_m\}$, say. Now $k_p \leq n + 2 \leq 2n - 2$. Hence, there exist P_i, P_j such that $P_i, P_j \notin (x_p)$. Then $Q_1 = \{P_1, P_i\}$, $Q_2 = \{P_2, P_j\}$ are two disjoint two subsets of P with $Q_1) \cup (Q_2) \subseteq \{x_m\}$. But this contradicts the fact that property $T(2)$ holds in D. Hence, $(P_1) \cap (P_2) = \{x_l\}$ for some $x_l \in B$.

Let $(P_1) = \{x_l, x_m\}$ and $(P_2) = \{x_l, x_{m'}\}$. Now $k_m, k_{m'} \leq n + 2 \leq 2n - 2$. Hence, there is a subscriber $P_i \neq P_2$ with $P_i \notin (x_m)$ and a subscriber $P_j \neq P_1$ with $P_j \notin (x_{m'})$. If $P_i \neq P_j$ then $Q_i = \{P_1, P_i\}$, $Q_2 = \{P_2, P_j\}$, are two disjoint two subsets of P such that $(Q_1) \cup (Q_2) \subseteq \{x_l\}$. Hence, since property $T(2)$ holds we must have $P_i = P_j$ and furthermore $B \setminus (x_m) = \{P_2, P_i\}$ and $B \setminus \{x_{m'}\} = \{P_1, P_i\}$. It follows that $k_m = k_{m'} = n + 2 = 2n - 2$. Hence, $n = 4$. Moreover $r_i = 2$ and $(P_i) = \{x_l, x_{m''}\}$ with $m'' \neq m, m'$. The arguments given above also show that $k_{m''} = n + 2$ and so $k_l = n + 1$, that is $x_l = x_4$. Now $B \setminus (x_{m''}) = \{P_1, P_2\}$ so that $(P_j) \supseteq \{x_1, x_2, x_3\}$ for $P_j \neq P_1, P_2, P_i$. Hence, $r_j \geq 3$ for $j \neq 1, 2, i$ and so $P_i = P_3$.

It follows from the previous lemma that a minimal RSN(4) with $r_i = 2$ for more than one $P_i \in P$ must be isomorphic to a SN(4) with incidence matrix

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Since $k_1 = k_2 = k_3 = n + 2$ and $k_4 = n + 1$ in this SN(4), by lemmas 1 and 2, properties $T(3)$ and $T(4)$ hold. Also, as $(P_i) \cap (P_j) \neq \Phi$ for all P_i, P_j, property $T(1)$ holds. Let Q_1 and Q_2 be disjoint two subsets of P. Since at least one element of $Q_1 \cup Q_2$ does not belong to $\{P_1, P_2, P_3\}$, $(Q_1) \cup (Q_2)$ contains at least one of x_1, x_2, x_3. If either Q_1 or Q_2 is contained in $\{P_1, P_2, P_3\}$ then $x_4 \in (Q_1) \cup (Q_2)$, otherwise $(Q_1) \cup (Q_2)$ contains at least two of x_1, x_2, x_3. In either case, $|(Q_1) \cup (Q_2)| \geq 2$ and property $T(2)$ holds. Thus, the matrix A_1 is the incidence matrix of a minimal RSN(4) by Proposition 1.

Suppose $D = (P, B, S)$ is a minimal RSN(4) in standard form with $r_1 = 2$ and $r_i > 2$ for $2 \leq i \leq 8$. Since $r_i + \cdots + r_8 = 23$ we have $r_i = 3$ for $i = 2, \cdots, 8$. Thus, an incidence matrix for D has one zero in each of rows 2–8. There are two possibilities according as $(P_1, x_4) \in S$ or $(P_1, x_4) \notin S$. In the first case, D is isomorphic to a SN(4) with incidence matrix

$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

In the second case D is isomorphic to a SN(4) with incidence matrix

$$A_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

An SN(4) with incidence matrix A_2 has $k_1 = k_2 = k_3 = 6$ and $k_4 = 5$. Hence, by Lemmas 1 and 2, properties $T(3)$ and $T(4)$ hold. As $(P_i) \cap (P_j) \neq \Phi$ for all P_i and P_j, property $T(1)$ holds.

Now for any two subset Q of P, $|(Q)| \geq 2$ unless $P_1 \in Q$. Hence, for any two disjoint two subsets Q_1 and Q_2 of P $|(Q) \cup (Q_2)| \geq 2$. Thus, property $T(2)$ also holds. Now by Proposition 1 a SN(4) with matrix A_2 is a minimal RSN(4).

It is easy to check that a SN(4) with incidence matrix A_3 is isomorphic to $M_1(4)$ and to $M_2(4)$ and so is a minimal RSN(4).

Theorem 3: Let $D = (P, B, S)$ be a minimal RSN(4). Then D is isomorphic to one of the minimal RSN(4) with incidence matrices A_1, A_2, A_3 given above.

Proof: We have $|S| = 23$. By Lemma 3, $r_i \geq 2$ for all $P_i \in P$. Since $|S| = r_1 + \cdots + r_8$ we must have $r_i = 2$ for some P_i. The result follows.

References

[1] R. A. Newbury and P. S. Raby, "Switch reduction in small telephone systems," *Electron. Lett.*, vol. 22, pp. 69–70, 1986.

[2] R. A. Newbury, "Minimum switches and connection procedure for eight telephone lines, *Electron. Lett.*, vol. 22, pp. 274–275, 1986.

[3] D. R. Hughes and F. C. Piper, *Design Theory*. London, England: Cambridge University Press, 1985.

[4] P. Hall, "On representatives of subsets," *JLMS*, vol. 10, pp. 26–30, 1935.

[5] R. A. Brualdi, *Introductory Combinatorics*. New York: North Holland, 1977.

[6] A. Tucker, *Applied Combinatorics*, 2nd ed. New York: Wiley, 1984.

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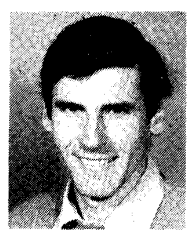

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