CHAPTER 4 - SOME CONSTRUCTIONS OF DIVISIBLE DESIGNS

In this Chapter we give two construction methods for strongly divisible 1- and 2-designs. They are both of a recursive nature, and may be used to construct many more structures and designs than those indicated.

In Section 4.1 a basic construction process for 1-structures admitting P Divisions is given, and we indicate under what conditions more specialised classes of structures and designs can be obtained from this method. In the second section we also consider a construction method, this one due originally to Sillitto, [46]. We again indicate when more specialised classes of structures can be derived.

Sections 4.3 and 4.4 give examples of the uses of the construction methods outlined in the first two sections. In Section 4.3 we first construct an infinite family of symmetric GD designs using the process of 4.2, and then use them, in conjunction with affine designs of appropriate orders, to obtain GD designs admitting strong tactical divisions using the method of 4.1. Under certain conditions these 1-designs are 2-designs.

In the fourth section we obtain a family of symmetric GD designs from affine geometries, and then we use these designs in the method of 4.1 (together with affine designs of appropriate parameters) to construct another family of 1-designs admitting strong tactical divisions. Again these 1-designs are 2-designs if the affine designs are affine planes, and in fact the 2-designs are Quasi-residual. We then show that these Quasi-residual designs are embeddable, and hence establish the existence of an infinite family of symmetric 2-designs.

Finally in Section 4.5 we indicate how the construction method of 4.1 may be modified to obtain the affine designs of Kimberley, [31].

In our examples of the use of the construction of Section 4.1 we have mainly restricted ourselves to cases in which the structure obtained is a 2-design. Clearly this construction process can yield many more 1-designs admitting P Divisions. For example we could use the strongly divisible 2-$(4m^2s^2, ms(2ms-1), (m+1)s(ms-1))$ designs of Theorem 4.3.3 in conjunction with affine planes of order m^2 to obtain further GD designs admitting strong tactical divisions.

The results of Sections 4.1, 4.2, 4.3 and 4.5 come from collaboration with H.J. Beker, and a slightly less general account may be found in [8]. Note that in [8] the construction method of 4.1 is described in terms of matrices; this approach is equivalent to that used in 4.1.

4.1 A Construction Method

In this section we first give a general, recursive method of construction for 1-structures, and then indicate under what conditions more specialised classes of structures and designs can be obtained from this construction method.

Suppose that $\underline{A}$ is a 1-(v,k,r) structure with b blocks admitting a parallelism with m blocks in each block class; (hence $v=mk$ and $b=mr$). Let $\underline{S}$ be a 1-(v',k',r') structure with b' blocks such that $v'=md$ for some integer d.

Let $\underline{A}_1,\dots,\underline{A}_d$ be d isomorphic copies of $\underline{A}$, and denote the points of $\underline{A}_i$ by X_{ij} $(1\leq j\leq v)$. Label the parallel classes of $\underline{A}_i$: $\underline{A}_{i1},\dots,\underline{A}_{ir}$ and the blocks of $\underline{A}_{i\ell}$: $y_{i\ell 1},\dots,y_{i\ell m}$. Since $\underline{A}_1,\dots,\underline{A}_d$ are isomorphic copies of $\underline{A}$, suppose $X_{ij}\in y_{i\ell s}$ if and only if $X_{i'j}\in y_{i'\ell s}$ for every i,i' $(1\leq i,i'\leq d)$.

Let $q_1,\dots,q_{b'}$ be a labelling of the blocks of $\underline{S}$. Suppose $\underline{S}_1,\dots,\underline{S}_d$ is some partitioning of the points of $\underline{S}$ into d classes of m points each, and then denote the points of $\underline{S}_i$ by $P_{i1},\dots,P_{im}$.

We now define a new incidence structure $\underline{D}$ with point set $\{X_{ij} \mid 1\leq i\leq d, 1\leq j\leq v\}$, and with block set $\{z_{\ell w} = \bigcup_{P_{is}\in q_w} y_{i\ell s} \mid 1\leq w\leq b',$

$1\leq \ell\leq r\}$.

Theorem 4.1.1 $\underline{D}$ is a 1-(mkd,kk',rr') structure.

Proof By definition $\underline{D}$ has $vd=mkd$ points, and every block of $\underline{D}$ is the union of k' blocks of $\underline{A}_{1\ell}\cup\underline{A}_{2\ell}\cup\ldots\cup\underline{A}_{d\ell}$ for some fixed ℓ. Hence, as these blocks are all disjoint, every block is incident with kk' points of $\underline{D}$.

Pick an integer ℓ $(1\leq\ell\leq r)$, and then consider any point X_{ij} of $\underline{A}_i$. This point occurs in exactly one block, $y_{i\ell s}$, say, of $\underline{A}_{i\ell}$. If $z_{\ell w}$ is a block of $\underline{D}$, then $z_{\ell w}=\bigcup_{P_{is}\in q_w} y_{i\ell s}$,

and so $X_{ij}\in z_{\ell w}$ if and only if $X_{ij}\in y_{i\ell s_1}\cup y_{i\ell s_2}\cup\ldots\cup y_{i\ell s_t}$ where $s_1,s_2,\ldots,s_t$ are such that $P_{is_1},\ldots,P_{is_t}$ are the points of $\underline{S}_i$ incident with q_w. Hence $X_{ij}\in z_{\ell w}$ if and only if P_{is} is incident with q_w. Every point of $\underline{S}$ is incident with precisely r' blocks q_w, and so there exist precisely r' blocks $z_{\ell w}$ of $\underline{D}$ such that $X_{ij}\in z_{\ell w}$.

There are r choices for ℓ and so every point of $\underline{D}$ is incident with rr' blocks of $\underline{D}$. ¤

Throughout this section we will use $\underline{S}$,$\underline{A}$ and $\underline{D}$ to denote the structures considered above. We now consider under what conditions $\underline{D}$ is a design. Let $\mu_{\underline{A}}$ be the largest intersection number of $\underline{A}$.

Lemma 4.1.2 $\underline{D}$ is a design if :

(i) $k>\mu_{\underline{A}}(m-1)$ and no two points of $\underline{A}$ are incident with the same set of blocks; and

(ii) $\underline{S}$ is a design and no block of $\underline{S}$ contains all the points from a given class of the partition $\underline{S}_1,\ldots,\underline{S}_d$ of the points of $\underline{S}$.

Proof Consider any two blocks $z_{\ell w}, z_{\ell' w'}$ of $\underline{D}$. Suppose they are incident with precisely the same set of points. By definition we have $y_{i\ell s_1} \cup y_{i\ell s_2} \cup \ldots \cup y_{i\ell s_t} = y_{i\ell' s_1'} \cup y_{i\ell' s_2'} \cup \ldots \cup y_{i\ell' s_{t'}'}$, (where $P_{is_1},\ldots,P_{is_t}$ are the points of $\underline{S}_i$ incident with q_w and $P_{is_1'},\ldots,P_{is_{t'}'}$ are the points of $\underline{S}_i$ incident with $q_{w'}$) for every i $(1\leq i\leq d)$. Now $1\leq t,t'\leq m-1$ (by (ii) of the Lemma) and so, since $k>\mu_{\underline{A}}(m-1)$, $y_{i\ell s_1} \cup y_{i\ell s_2} \cup \ldots \cup y_{i\ell s_t}$ can only contain all k points of $y_{i\ell' s_1'}$ if $\ell=\ell'$. Hence $\{s_1,\ldots,s_t\} = \{s_1',\ldots,s_{t'}'\}$, i.e. q_w and $q_{w'}$ are incident with the same set of points of $\underline{S}_i$. This is true for every i $(1\leq i\leq d)$, and so, since $\underline{S}$ is a design, $w=w'$. Hence $z_{\ell w} = z_{\ell' w'}$ and no two blocks of $\underline{D}$ are incident with the same point set.

Suppose X_{ij} and $X_{i'j'}$ are two points of $\underline{D}$ incident with the same set of blocks of $\underline{D}$. Let X_{ij} be incident with the blocks $y_{i1s_1}, y_{i2s_2},\ldots,y_{irs_r}$ of $\underline{A}_i$ and $X_{i'j'}$ be incident with the blocks

$y_{i'1s'_1}, y_{i'2s'_2}, \ldots, y_{i'rs'_r}$ of $\underline{A}_{i'}$. Now X_{ij} and $X_{i'j'}$ are incident with the block $z_{\ell w}$ of $\underline{D}$ if and only if $P_{is_\ell} \in q_w$ and $P_{i's'_\ell} \in q_w$. So P_{is_ℓ} and $P_{i's'_\ell}$ are incident with the same set of blocks of $\underline{S}$, and hence, since $\underline{S}$ is a design, $i=i'$ and $s_\ell = s'_\ell$ for every ℓ. So X_{ij} and $X_{ij'}$ are incident with the same set of blocks of $\underline{A}_i$, and so, by (i) above, $j=j'$. Hence no two points of $\underline{D}$ can be incident with the same set of blocks of $\underline{D}$. ¤

Remarks Note that (i) of the above Lemma implies that $\underline{A}$ is a design. Also, if $\underline{A}$ is an affine 2-design, then (by Result 1.5.5) its parameters may be written $2\text{-}(\mu m^2, \mu m, (\mu m-1)/(m-1))$, and thus, since $\mu = \mu_{\underline{A}}$, (i) above is automatically satisfied.

Theorem 4.1.3 Suppose that $\underline{A}$ is a $2\text{-}(v,k,\lambda)$ structure and $\underline{S}$ admits a point division $\underline{S}_1, \ldots, \underline{S}_d$ with $|\underline{S}_i| = m$ for every i $(1 \leq i \leq d)$ and connection numbers λ'_{ij}. Then $\underline{D}$, constructed using this partition of the points of $\underline{S}$, admits a P Division $\underline{P}_1, \ldots, \underline{P}_d$, where $\underline{P}_i$ contains the points of $\underline{A}_i$, (and hence $|\underline{P}_i| = v = mk$), and $\underline{D}$ has connection numbers.

$$r\lambda'_{ij} + \lambda(r' - \lambda'_{ii})\delta_{ij}, \quad (1 \leq i,j \leq d).$$

Proof Suppose X_{ij} and $X_{i'j'}$ are two distinct points of $\underline{D}$. There are two cases to consider; we first suppose $i=i'$.

$\underline{A}_i$ is a $2\text{-}(v,k,\lambda)$ structure with r parallel classes, and so in precisely λ of these classes X_{ij} and $X_{ij'}$ occur together on a block. Pick such a parallel class, $\underline{A}_{i\ell}$ say, and suppose $y_{i\ell s}$ is the block of $\underline{A}_{i\ell}$ incident with both X_{ij} and $X_{ij'}$.

Consider any block $z_{\ell w}$ of $\underline{D}$. Then X_{ij}, $X_{ij'} \in z_{\ell w}$ if and only if $X_{ij}, X_{ij'} \in y_{i\ell s_1} \cup \ldots \cup y_{i\ell s_t}$, where $P_{is_1}, \ldots, P_{is_t}$ are the points of $\underline{S}_i$ incident with q_w. This occurs if and only if $s \in \{s_1, \ldots, s_t\}$, i.e. if and only if P_{is} is incident with q_w. But every point of $\underline{S}$ is incident with precisely r' blocks q_w, and so there are exactly r' blocks $z_{\ell w}$ with $X_{ij}, X_{ij'} \in z_{\ell w}$ for every ℓ satisfying the above condition.

Now consider the other $r-\lambda$ parallel classes of $\underline{A}_i$, in which X_{ij} and $X_{ij'}$ occur in different blocks. Let $\underline{A}_{i\ell}$ be such a parallel class of $\underline{A}_i$, and suppose $y_{i\ell s}$ and $y_{i\ell s'}$ are the blocks of $\underline{A}_i$ which contain X_{ij} and $X_{ij'}$, respectively. Suppose $z_{\ell w}$ is a block of $\underline{D}$, and then $X_{ij}, X_{ij'} \in z_{\ell w}$ if and only if

$X_{ij}, X_{ij'} \in y_{i\ell s_1} \cup y_{i\ell s_2} \cup \ldots \cup y_{i\ell s_t}$ where $P_{is_1}, \ldots, P_{is_t}$ are the points of $\underline{S}_i$ incident with q_w. This occurs if and only if $s, s' \in \{s_1, \ldots, s_t\}$, i.e. if and only if P_{is} and $P_{is'}$ are incident with q_w, and there are precisely λ'_{ii} blocks of $\underline{S}$ incident with any two points of $\underline{S}_i$. So there are λ'_{ii} blocks $z_{\ell w}$ incident with X_{ij} and $X_{ij'}$ where ℓ is of the above type.

Summing over all possible ℓ we see that X_{ij} and $X_{ij'}$ are incident with precisely $\lambda r' + (r-\lambda)\lambda'_{ii}$ blocks of $\underline{D}$.

Now suppose $i \neq i'$. Pick some ℓ, $(1 \leq \ell \leq r)$, and then X_{ij} and $X_{i'j'}$ are incident with precisely one block of $\underline{A}_{i\ell}$ and $\underline{A}_{i'\ell}$, respectively. Label these blocks $y_{i\ell s}$ and $y_{i'\ell s'}$. Suppose $z_{\ell w}$ is a block of $\underline{D}$, then $X_{ij}, X_{i'j'} \in z_{\ell w}$ if and only if $X_{ij} \in y_{i\ell s_1} \cup \ldots \cup y_{i\ell s_t}$ and $X_{i'j'} \in y_{i'\ell s'_1} \cup \ldots \cup y_{i'\ell s'_{t'}}$ where $P_{is_1}, \ldots, P_{is_t}$ are the points of $\underline{S}_i$ incident with q_w and $P_{i's'_1}, \ldots, P_{i's'_{t'}}$ are the points of $\underline{S}_{i'}$ incident with q_w. This occurs if and only if $s \in \{s_1, \ldots, s_t\}$ and $s' \in \{s'_1, \ldots, s'_{t'}\}$, i.e. if and only if both P_{is} and $P_{i's'}$ are incident with q_w. Since $\underline{S}_1, \ldots, \underline{S}_d$ forms a P Division of $\underline{S}$, there are precisely $\lambda_{ii'}$ such blocks q_w, and hence, summing over all ℓ $(1 \leq \ell \leq r)$, we see that there are $r\lambda'_{ii}$ blocks of $\underline{D}$ incident with both X_{ij} and $X_{i'j'}$. ◻

From now on we will assume that $\underline{S}$, $\underline{A}$ and $\underline{D}$ are as in Theorem 4.1.3.

<u>Lemma 4.1.4</u>

(i) $\underline{S}_1, \ldots, \underline{S}_d$ is a CLP Division of $\underline{S}$ if and only if $\underline{P}_1, \ldots, \underline{P}_d$ is a CLP Division of $\underline{D}$.

(ii) Every block of $\underline{S}$ contains equally many points of $\underline{S}_i$ if and only if every block of $\underline{D}$ contains equally many points of $\underline{P}_i$.

(iii) $\underline{S}_1,\ldots,\underline{S}_d$ is a group division of $\underline{S}$ or $\underline{S}$ is a 2-structure if and only if $\underline{P}_1,\ldots,\underline{P}_d$ is a group division of $\underline{D}$ or $\underline{D}$ is a 2-structure.

(iv) $\underline{S}_1,\ldots,\underline{S}_d$ is a semi-regular group division of $\underline{S}$ if and only if $\underline{P}_1,\ldots,\underline{P}_d$ is a semi-regular group division of $\underline{D}$.

(v) $\underline{D}$ is a 2-structure if and only if $\underline{S}_1,\ldots,\underline{S}_d$ is a group division of $\underline{S}$ satisfying $r\lambda'_1+\lambda(r'-\lambda'_1)=r\lambda'_2$ where $\lambda'_1=\lambda'_{ii}$, $\lambda'_2=\lambda'_{ij}$ for every i,j $(1\leq i,j\leq d, i\neq j)$.

Proof (i) $\underline{S}_1,\ldots,\underline{S}_d$ is a CLP Division if and only if there exist a λ' with $\lambda'_{ii}=\lambda'$ for every i $(1\leq i\leq d)$, i.e. if and only if $\lambda r'+(r-\lambda)\lambda'_{ii}=\lambda r'+(r-\lambda)\lambda'$ for every i, i.e. if and only if $\underline{P}_1,\ldots,\underline{P}_d$ is a CLP Division.

(ii) If $z_{\ell w}$ is a block of $\underline{D}$, then $|z_{\ell w}\cap \underline{P}_i| = |y_{i\ell s_1}\cup\ldots\cup y_{i\ell s_t}|$ (where $P_{is_1},\ldots,P_{is_t}$ are the blocks of $\underline{S}_i$ incident with q_w) $= k|q_w\cap \underline{S}_i|$; and (ii) follows.

(iii) $\underline{S}_1,\ldots,\underline{S}_d$ is a group division or $\underline{S}$ is a 2-structure if and only if there exist λ'_1,λ'_2 with $\lambda'_{ii}=\lambda'_1$, $\lambda'_{ij}=\lambda'_2$ for every i,j $(i\neq j)$, i.e. if and only if $r\lambda'_{ij}+\lambda(r'-\lambda'_{ii})\delta_{ij}=r\lambda'_2+\lambda(r'-\lambda'_1)\delta_{ij}$ for every i,j $(i\neq j)$, i.e. if and only if $\underline{P}_1,\ldots,\underline{P}_d$ is a group division or $\underline{D}$ is a 2-structure.

(iv) Immediate for (ii),(iii) above and Result 1.4.6.

(v) Immediate from (iii) above. ¤

<u>Remarks</u> From (ii) we can derive the result that $\underline{S}_1,\ldots,\underline{S}_d$ is a SRP Division of $\underline{S}$ if and only if $\underline{P}_1,\ldots,\underline{P}_d$ is a SRP Division of $\underline{D}$. But since $\underline{S}_1,\ldots,\underline{S}_d$ has constant class size, this says no more than (iv), (using Theorem 2.3.5).

From (iii) we may also deduce that if $\underline{D}$ is a 2-design then $\underline{S}_1,\ldots,\underline{S}_d$ is a group division, and, using the above notation $\lambda'_1<\lambda'_2$.

<u>Lemma 4.1.5</u> Suppose $\underline{S}$ admits a tactical division $T(\underline{S})$ with point classes $\underline{S}_1,\ldots,\underline{S}_d$ and c block classes.

Then $\underline{D}$ admits a tactical division $T(\underline{D})$ with point classes $\underline{P}_1,\ldots,\underline{P}_d$ and rc block classes. Also, $T(\underline{D})$ is strong if and only if $r(b'-c)=d(mk-1)$.

Proof By 4.1.4(i), $\underline{P}_1,\ldots,\underline{P}_d$ is a CLP Division. Let $\underline{C}_1,\ldots,\underline{C}_c$ be a labelling of the block classes of $T(\underline{S})$, and denote the number of blocks of $\underline{C}_j$ incident with a point of $\underline{S}_i$, and the number of points of $\underline{S}_i$ incident with a block of $\underline{C}_j$ by γ_{ij} and β_{ij} respectively. Further, suppose $|\underline{C}_j| = m_j$, $(1\le j\le c)$.

Relabel the blocks of $\underline{S}$: q_{ja} $(1\le j\le c, 1\le a\le m_j)$, so that $\underline{C}_j = \{q_{ja} | 1\le a\le m_j\}$ and then relabel the blocks of $\underline{D}$: $z_{\ell ja} = \bigcup_{P_{is}\in q_{ja}} y_{i\ell s}$, $(1\le \ell\le r, 1\le j\le c, 1\le a\le m_j)$. Finally set

$\underline{B}_{\ell j} = \{z_{\ell ja} | 1\le a\le m_j\}$ for every ℓ, j $(1\le\ell\le r, 1\le j\le c)$, and we have a partition of the blocks of $\underline{D}$ into rc classes. We now show that $T(\underline{D})$, defined as having point classes $\underline{P}_1,\ldots,\underline{P}_d$ and block classes $\underline{B}_{\ell j}$ forms a tactical decomposition of $\underline{D}$.

Pick some point class $\underline{P}_i$ and a block class $\underline{B}_{\ell j}$. Then if $X_{iw}\in\underline{P}_i$, there exists a unique block, $y_{i\ell s}$ say, of $\underline{A}_{i\ell}$ which is incident with X_{iw}. So X_{iw} is incident with a block $z_{\ell ja}\in\underline{B}_{\ell j}$ if and only if $P_{is}\in q_{ja}$ (by similar arguments to those used above). But P_{is} is incident with γ_{ij} blocks of the form q_{ja} $(1\le a\le m_j)$, and so any point of $\underline{P}_i$ is incident with precisely γ_{ij} blocks of $\underline{B}_{\ell j}$.

Now let $z_{\ell ja}$ be a block of $\underline{B}_{\ell j}$. By definition $z_{\ell ja} = \bigcup_{P_{is}\in q_{ja}} y_{i\ell s}$, and so the number of points of $\underline{P}_i$ incident with $z_{\ell ja}=|y_{i\ell s_1}\cup\ldots\cup y_{i\ell s_t}|$, (where $P_{is_1},\ldots,P_{is_t}$ are the points of $\underline{S}_i$ incident with q_{ja}), $= kt = k\ \beta_{ij}$.

So $T(\underline{D})$ is a tactical division, which is strong if and only if "$b+d=v+c$", i.e. if and only if $b'r+d=mkd+rc$. ⌑

We may now state :

<u>Theorem 4.1.6</u> If $\underline{S}$,$\underline{A}$ and $\underline{D}$ are as in Lemma 4.1.5 then any two of the following imply the third :

(i) $\underline{A}$ is affine;

(ii) $T(\underline{S})$ is strong;

(iii) $T(\underline{D})$ is strong.

<u>Proof</u> Suppose (i) holds, i.e. suppose $\underline{D}$ is a $2\text{-}(\mu m^2,\mu m,(\mu m-1)/(m-1))$ design. $T(\underline{S})$ is strong if and only if $b'+d=md+c$, i.e. if and only if $rb'-rc=rd(m-1)=(\mu m^2-1)d=mkd-d$, i.e. if and only if $T(\underline{D})$ is strong, (by Lemma 4.1.5). Hence (i) and (ii) hold if and only if (i) and (iii) hold.

Suppose (ii) holds, i.e. suppose $b'+d=md+c$. Then $T(\underline{D})$ is strong if and only if $d(mk-1)=r(b'-c)=rd(m-1)$, i.e. if and only if (i) holds, (using Result 1.5.5). Hence (ii) and (iii) hold if and only if (i) and (ii) hold. ⌑

[redacted]

We now turn our attention to the intersection numbers of $\underline{D}$. If $\underline{S}$,$\underline{A}$ and $\underline{D}$ are as in Lemma 4.1.5 then (by Result 1.5.2) if $\underline{S}$ is a design and T($\underline{S}$) is strong the intersection number of a block of class $\underline{C}_i$ and a block of class $\underline{C}_j$ depends only on i and j, and in this case we denote it by ρ'_{ij}, $(1\leq i,j\leq c)$. Further $\rho'_{jj}=\rho'=k'-r'+\lambda'_1$ for every i $(1\leq j\leq c)$, where $\lambda'_1=\lambda'_{ii}$ for every i $(1\leq i\leq d)$.

Now consider $\underline{D}$. Relabel the block classes of T($\underline{D}$): $\underline{Y}_1,\ldots,\underline{Y}_{rc}$ where $\underline{Y}_i=\underline{B}_{\ell j}$ if $i=(\ell-1)c+j$. Now suppose that $\underline{D}$ is a design and T($\underline{D}$) is strong. Again by Result 1.5.2, we denote the intersection number of a block of class $\underline{Y}_i$ and a block of class $\underline{Y}_j$ by ρ_{ij} $(1\leq i,j\leq rc)$, and we also have $\rho=\rho_{ii}$ $(1\leq i\leq rc)$.

<u>Theorem 4.1.7</u> Suppose $\underline{S}$,$\underline{A}$ and $\underline{D}$ are as in Lemma 4.1.5, and also suppose $\underline{S}$ and $\underline{D}$ are designs. If $\underline{A}$ is an affine design and T($\underline{S}$) is strong, then T($\underline{D}$) is strong, and the intersection numbers of $\underline{D}$ satisfy :

$$\rho_{ij} = k\rho'_{uw} \qquad 1\leq i,j\leq rc;\ 1\leq u,w\leq c;\quad u\equiv i\not\equiv j\equiv w \pmod c.$$

$$\rho_{ij} = \left(\rho'+(r'-\lambda')/m_t\right)k \qquad 1\leq i,j\leq rc:\quad \underset{i\neq j}{i\equiv j\equiv t \pmod c};\ 1\leq t\leq c;$$

$$\rho = k\rho'.$$

Proof Suppose $\underline{Y}_i$ and $\underline{Y}_{i'}$ are two block classes of $T(\underline{D})$, where $i = (\ell-1)c+j$ and $i' = (\ell'-1)c+j'$.

If $\ell=\ell'$ and $z_{\ell ja}\in\underline{Y}_i$, $z_{\ell j'a'}\in\underline{Y}_{i'}$, then :

$$|z_{\ell ja}\cap z_{\ell j'a'}| = \Big|\bigcup_{P_{is}\in q_{ja}} y_{i\ell s} \cap \bigcup_{P_{is'}\in q_{j'a'}} y_{i\ell s'}\Big|$$

$$= \sum_{i=1}^{d} |(y_{i\ell s_1}\cup\ldots\cup y_{i\ell s_t}) \cap (y_{i\ell s'_1}\cup\ldots\cup y_{i\ell s'_{t'}})|$$

(where $P_{is_1},\ldots,P_{is_t}$ are the points of $\underline{S}_i$ incident with q_{ja}, and $P_{is'_1},\ldots,P_{is'_{t'}}$ are the points of $\underline{S}_i$ incident with $q_{j'a'}$).

$$= k\sum_{i=1}^{d} |\{s_1,\ldots,s_t\} \cap \{s'_1,\ldots,s'_{t'}\}|$$

$$= k\rho'_{jj'} \qquad (\underline{1}).$$

If $\ell\neq\ell'$ and $z_{\ell ja}\in\underline{Y}_i$, $z_{\ell' j'a'}\in\underline{Y}_{i'}$, then :

$$|z_{\ell ja}\cap z_{\ell' j'a'}| = \sum_{i=1}^{d} |(y_{i\ell s_1}\cup\ldots y_{i\ell s_t}) \cap (y_{i\ell' s'_1}\cup\ldots\cup y_{i\ell' s'_{t'}})|$$

(where $P_{is},\ldots,P_{is_t}$ and $P_{is'_1},\ldots,P_{is'_{t'}}$ are the points of $\underline{S}_i$ incident with q_{ja} and $q_{j'a'}$ respectively).

$$= \sum_{i=1}^{d} \sum_{u=1}^{t} \sum_{w=1}^{t'} |y_{i\ell s_u} \cap y_{i\ell' s'_w}|$$

$$= \sum_{i=1}^{d} tt'\mu$$

$$= \mu \sum_{i=1}^{d} \beta'_{ij} \beta'_{ij'}$$

$$= \mu m(\rho'_{jj'} + \delta_{jj'}(k' - \rho_{jj'})/m_j) \qquad \text{(by Lemma 2.4.2(vi))}$$

$$= k(\rho'_{jj'} + \delta_{jj'}(k' - \rho_{jj'})/m_j) \qquad (\underline{2}).$$

Combining (1) and (2) the theorem follows. ¤

Remark By Theorem 4.1.7, if $\underline{S}$ has i intersection numbers, then $\underline{D}$ has at most i+s, where s is the number of distinct values of m_j $(1 \leq j \leq c)$.

4.2 A Construction Method of Sillitto

In this section we examine a method of construction due to Sillitto, [46], and show that it may be used to construct designs admitting strong tactical decompositions. Suppose $\underline{E}$ and $\underline{F}$ are a 1-$(\bar{v},\bar{k},\bar{r})$ structure and a 1-(v,k,r) structure with incidence matrices A and B, respectively. Then let $C = A \otimes B + (J-A) \otimes (J-B)$ where $\otimes$ denotes Kronecker product. Since A and B are 0,1 matrices, C is a 0,1 matrix, and so C is an incidence matrix of a structure $\underline{S}$. We then immediately have :

<u>Theorem 4.2.1</u> $\underline{S}$ is a $1-\left(v\bar{v}, k\bar{k}+(v-k)(\bar{v}-\bar{k}), r\bar{r}+(b-r)(\bar{b}-\bar{r})\right)$ structure with $b\bar{b}$ blocks.

Throughout this section we assume that $\underline{E}$, $\underline{F}$ and $\underline{S}$ are as defined above. Another immediate result is :

<u>Lemma 4.2.2</u> If $v \neq 2k$ or $\bar{v} \neq 2\bar{k}$, then $\underline{S}$ is a design if and only if $\underline{E}$ and $\underline{F}$ are designs.

<u>Remark</u> If $\underline{S}$ is a design then $\underline{E}$ and $\underline{F}$ are always designs, but the reverse is not true if $v=2k$ and $\bar{v}=2\bar{k}$. Suppose $\underline{E}$, $\underline{F}$ are both the trivial 2-(4,2,1) design with

$$A = B = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} .$$

Then the first and eigth columns of $C = A \otimes B + (J-A) \otimes (J-B)$ are identical, i.e. $\underline{S}$ is not a design.

In fact the Lemma fails if and only if $\underline{E}$ or $\underline{F}$ satisfy $\bar{v}=2\bar{k}$ and $v=2k$, and there exists a pair of points (or blocks) in both $\underline{E}$ and $\underline{F}$ having connection (or intersection) number zero.

We can now consider partitions of the points (or blocks) of $\underline{F}$ which give rise to partitions of the points (or blocks) of $\underline{S}$. Suppose $\underline{P}_1, \ldots, \underline{P}_d$ is a partition of the points of $\underline{F}$. By the nature of the construction for $\underline{S}$, for any i ($i \leq i \leq \bar{v}$) the

the $(i-1)v+1,(i-1)v+2,\ldots,iv$ rows of C are indexed by the rows of B, and hence by the points of $\underline{F}$. So $\underline{P}_1,\ldots,\underline{P}_d$ induces a partition of these rows into d classes, and repeating this process for every i we obtain a partition of the rows of C (and hence of the points of $\underline{S}$) into $\overline{v}d$ classes. Label the classes from rows $(i-1)v+1,(i-1)v+2,\ldots,iv$ (corresponding to $\underline{P}_1,\ldots,\underline{P}_d$) $\underline{S}_{(i-1)d+1},\underline{S}_{(i-1)d+2},\ldots,\underline{S}_{id}$, and then $\underline{S}_1,\ldots,\underline{S}_{\overline{v}d}$ partitions the points of $\underline{S}$. We call this partition the partition <u>induced by</u> $\underline{P}_1,\ldots,\underline{P}_d$. We make similar definitions for partitions of blocks of $\underline{F}$ and hence $\underline{S}$, and so if $T(\underline{F})$ is a tactical decomposition of $\underline{F}$, then we will refer to the partition $T(\underline{S})$ of the points and blocks of $\underline{S}$ as the induced partition of $\underline{S}$. Note that we will always denote any induced partition of the points of $\underline{S}$ by $\underline{S}_1,\ldots,\underline{S}_{\overline{v}d}$ and any induced partition of the blocks of $\underline{S}$ by $\underline{U}_1,\ldots,\underline{U}_{\overline{b}c}$.

<u>Theorem 4.2.3</u> Suppose $\underline{F}$ admits a P Division $\underline{F}_1,\ldots,\underline{F}_d$ with connection numbers λ_{ij}, $(1\leq i,j\leq d)$, and $\underline{E}$ is a 2-$(\overline{v},\overline{k},\overline{\lambda})$ structure with $\overline{b} = 4(\overline{r}-\overline{\lambda})$. Then the partition $\underline{S}_1,\ldots,\underline{S}_{\overline{v}d}$ of the points of $\underline{S}$ induced by $\underline{F}_1,\ldots,\underline{F}_d$ is a P Division, and in this case a point of class $\underline{S}_u$ and a point of class $\underline{S}_w$ (where $u = (f-1)v+i$ and $w = (g-1)v+j$) have connection number :-

$$2(\overline{r}-2\overline{\lambda})(b-r)+ \overline{\lambda}b + \delta_{fg}(\overline{r}-\overline{\lambda})\big(b-4(r-\lambda_{ij})\big).$$

Proof Using the fact that $\underline{E}$ is a 2-$(\bar{v},\bar{k},\bar{\lambda})$ structure, computation yields : $CC^T = I_{\bar{v}} \otimes [(\bar{b}-\bar{r})(b-2r)J_v+\bar{b}BB^T] +$

$$(J_{\bar{v}} - I_{\bar{v}})\otimes\{[2(\bar{r}-\bar{\lambda})r+(\bar{b}-2\bar{r}+\bar{\lambda})(b-2r)]J_v+[\bar{b}-4(\bar{r}-\bar{\lambda})BB^T]\}.$$

Hence we see that $\underline{S}_1,\ldots,\underline{S}_{\bar{v}d}$ is a P Division if and only if $\bar{b} = 4(\bar{r}-\bar{\lambda})$. Assuming this and manipulating the above expression, the Theorem follows. ¤

Remark If $\underline{D}$ is an arbitrary 2-(v,k,λ) structure we will say that $\underline{D}$ satisfies Condition * if $b = 4(r-\lambda)$.

Given that $\underline{E}$ and $\underline{F}$ satisfy the condition of Theorem 4.2.3 we have the following immediate corollaries :

Corollary 4.2.4

(i) If $\underline{F}_1,\ldots,\underline{F}_d$ is a CLP Division, then so is $\underline{S}_1,\ldots,\underline{S}_{\bar{v}d}$.

(ii) If $\underline{F}_1,\ldots,\underline{F}_d$ is a Group Division with connection numbers λ,λ', then $\underline{S}_1,\ldots,\underline{S}_{\bar{v}d}$ is a Group Division if and only if $b = 4(r-\lambda')$, and in this case the connection number of a point of class $\underline{S}_u$ and a point of class $\underline{S}_w$ is $2(\bar{r}-2\bar{\lambda})(b-r)+\bar{\lambda}b+\delta_{uw}(\bar{r}-\bar{\lambda})\{b-4(r-\lambda)\}$.

Corollary 4.2.5 If $\underline{F}$ is a 2-(v,k,λ) structure and we consider the P Division of $\underline{F}$ with one class, consisting of all the points of $\underline{F}$, then the induced P Division

$\underline{S}_1,\ldots,\underline{S}_{\bar{v}}$ of $\underline{S}$ is a group division, and the connection number of a point of class $\underline{S}_u$ and a point of class $\underline{S}_w$ =

$$2(\bar{r}-2\bar{\lambda})(b-r)+\bar{\lambda}b+\delta_{uw}(\bar{r}-\bar{\lambda})(b-4(r-\lambda)).$$

Corollary 4.2.6 (c.f. [42], Theorem 1 and [46])

If $\underline{F}$ is a 2-(v,k,λ) structure then $\underline{S}$ is a 2-structure if and only if $\underline{F}$ satisfies Condition *. In this case $\underline{S}$ is a

$$2\text{-}\big(v\bar{v},k\bar{k}+(v-k)(\bar{v}-\bar{k}),2(\bar{r}-2\bar{\lambda})(b-r)+\bar{\lambda}b\big)$$ structure, and $\underline{S}$ satisfies Condition *.

Proof By Corollary 4.2.5 we need only check that $\underline{S}$ satisfies Condition *. $\underline{S}$ has $\bar{b}b$ blocks and $\bar{r}r+(\bar{b}-\bar{r})(b-r)$ blocks incident with every point, and so "$4(r-\lambda)$" =

$$4\big(\bar{r}r+(\bar{b}-\bar{r})(b-r)-2(\bar{r}-2\bar{\lambda})(b-r)-\bar{\lambda}b\big) = b(4\bar{b}-12\bar{r}+12\bar{\lambda}),$$ using

the fact that $\bar{b}=4(\bar{r}-\bar{\lambda})$. Hence "$b-4(r-\lambda)$"$= b\bar{b}-b(4\bar{b}-12\bar{r}+12\bar{\lambda}) = 0$. ∎

Remark Corollary 4.2.6 gives a recursive method for the construction of 2-structures satisfying Condition *.

Next we consider structures $\underline{F}$ admitting tactical decompositions.

Lemma 4.2.7 If $\underline{F}$ admits a tactical decomposition $T(\underline{F})$ with d point classes and c block classes, then the induced partition $T(\underline{S})$ of the points and blocks of $\underline{S}$ is a tactical decomposition of $\underline{S}$ with $\bar{v}d$ point classes and $\bar{b}c$ block classes.

Proof T($\underline{S}$) clearly has the requisite number of point and block classes, and is a tactical decomposition by definition of C, since the submatrix of C corresponding to rows $(i-1)v+1,(i-1)v+2,\dots,iv$ and columns $(j-1)b+1,(j-1)b+2,\dots,jb$ $(1\le i\le\bar{v},1\le j\le\bar{b})$, is either B or J-B. ¤

Using the above results we now have :

Theorem 4.2.8 If T($\underline{F}$) is a tactical division of $\underline{F}$ with d point classes and c block classes, and $\underline{E}$ is a 2-$(\bar{v},\bar{k},\bar{\lambda})$ structure satisfying Condition *, then the induced decomposition T($\underline{S}$) is a tactical division of $\underline{S}$. In this case any two of the following imply the third :

(i) $\underline{E}$ is symmetric;

(ii) T($\underline{F}$) is strong;

(iii) T($\underline{S}$) is strong.

If $\underline{E}$ is symmetric, T($\underline{F}$) is strong and $\underline{F}$ has intersection numbers ρ_{ij} $(1\le i,j\le c)$, then T($\underline{S}$) is strong, and if the block classes of T($\underline{S}$) are $\underline{U}_1,\dots,\underline{U}_{\bar{b}c}$ the intersection number of a block of class $\underline{U}_u$ and a block of class $\underline{U}_w$ (where $u=(f-1)b+i$, $w=(g-1)b+j$) $= 2(\bar{r}-2\bar{\lambda})(v-k)+\bar{\lambda}v+\delta_{fg}(\bar{r}-\bar{\lambda})(v-4(k-\rho_{ij}))$.

Proof $T(\underline{S})$ is a tactical decomposition by Lemma 4.2.7, and is thus a tactical division of $\underline{S}$ by Corollary 4.2.4(i). $T(\underline{S})$ has $\bar{v}d$ point classes and $\bar{b}c$ block classes, and so $T(\underline{S})$ is strong if and only if $\bar{b}(b-c) = \bar{v}(v-d)$, and the second part of the theorem follows. Finally, the intersection numbers of $T(\underline{S})$ are as given, applying Theorem 4.2.3 to $\underline{S}^*$. ¤

Remarks If $\underline{F}$ is a design, then by Result 1.5.1, $b-c \geq v-d$ and so, using Result 1.2.5, (i) and (ii) above hold if and only if (iii) holds. The result on the intersection numbers of $\underline{S}$ generalises Theorem 5.1 of Kageyama, [27]. We may find the connection numbers of $\underline{S}$ by using Theorem 4.2.3.

Given that $\underline{E},\underline{F}$ satisfy the conditions of Theorem 4.2.8, we have :

Corollary 4.2.9 Suppose $\underline{F}$ is a 2-structure and let $T(\underline{F})$ be the tactical division of $\underline{F}$ with just one point class and one block class. Then the induced division $T(\underline{S})$ is a tactical division with $\bar{v}$ point classes and $\bar{b}$ block classes, and the point classes of $T(\underline{S})$ form a group division of $\underline{S}$. Furthermore $T(\underline{S})$ is strong if and only if both $\underline{E}$ and $\underline{F}$ are symmetric.

Proof $T(\underline{S})$ is a tactical division by Theorem 4.2.8; the point classes form a group division by Corollary 4.2.5, and by Result 1.2.5 the Corollary follows, since $T(\underline{S})$ is strong if and only if $\bar{b}(b-1) = \bar{v}(v-1)$. ¤

Finally we give some results of Shrikhande and Sillitto on the parameters of 2-structures satisfying Condition *.

Result 4.2.10 (Sillitto, [46]) $\underline{S}$ is a 2-structure satisfying Condition * if and only if $\underline{S}$ is a $2\text{-}\left(u^2, u(u\pm1)/2, (u\pm2)N/2\right)$ structure for some integers u,N.

Result 4.2.11 (Shrikhande, [42]) $\underline{S}$ is a symmetric 2-design satisfying Condition * if and only if $\underline{S}$ is a $2\text{-}\left(4s^2, s(2s\pm1), s(s\pm1)\right)$ design for some integer s.

Remark In these two results, the structures with the parameters corresponding to the + signs represent the complements of the structures with parameters corresponding to the - signs.

If $\underline{S}$ is a $2\text{-}\left(4s^2, s(2s-1), s(s-1)\right)$ design, then we say that $\underline{S}$ is a $S(s)$, and we denote the complementary designs by $S(s)^c$.

Result 4.2.12 ([42],[46])

(i) If a $S(m)$ and a $S(n)$ exist , then a $S(2mn)$ exists.

(ii) If e and e+2 are both odd prime powers, then a $S\left((e+1)/2\right)$ exists.

(iii) $S(1)$ exists.

Remarks (i) follows from Corollary 4.2.6 by observing that if $\underline{E}$ is a $S(m)$ and $\underline{F}$ is a $S(n)$ then $\underline{S}$ is a $S(2mn)^c$, and note that

$S(1)$ is the trivial "2-(4,1,0)" design for which I_4 is an incidence matrix. Using (i),(ii),(iii) above we may deduce that $S(m)$ exists for $1 \leq m \leq 10$ $(m \neq 7)$. In fact many constructions exist for $S(m)$ designs, and the smallest value of m for which the author knows of no $S(m)$ designs is $m = 11$.

It is also clear from the above that a $S(m)$ exists for infinitely many values of m.

4.3 A Family of Strongly Divisible Designs

In this section we utilize the construction methods of 4.1 and 4.2, to obtain an infinite family of group divisible 1-designs and 2-designs admitting strong tactical divisions.

Lemma 4.3.1 If $\underline{E}$ is a $S(s)^c$ and $\underline{F}$ is a symmetric $2\text{-}(m,h,\lambda)$ design, then $\underline{S}$ constructed as in Section 4.2 above is a symmetric GD $1\text{-}\left(4ms^2, ms(2s-1)+2hs, ms(2s-1)+2hs\right)$ design. $\underline{S}$ admits a strong tactical division $T(\underline{S})$ with $4s^2$ point and block classes of m points and blocks each (the point classes of $T(\underline{S})$ being the classes of the group division), and with connection numbers λ'_{ij} and intersection numbers ρ'_{ij}, where

$$\lambda'_{ij} = \rho'_{ij} = ms(s-1)+2hs+\delta_{ij}\ s^2\left(m-4(h-\lambda)\right); \quad 1 \leq i,j \leq m.$$

Proof By Corollary 4.2.9, if we let $T(\underline{F})$ be the tactical division of $\underline{F}$ with just one point class and one block class, then the induced division is a tactical division with $\bar{v} = \bar{b} = 4s^2$

point and block classes of m points and blocks each. The parameters of $\underline{S}$ can be obtained by applying Theorems 4.2.1, 4.2.3 and 4.2.8. Finally $\underline{S}$ is a design by Lemma 4.2.2. ¤

Remark We take $\underline{E} = S(s)^c$ and not $S(s)$ in the Lemma above so that the designs obtained have $k<v/2$.

Theorem 4.3.2 Let $\underline{A}$ be a $2\text{-}\left(\mu m^2, \mu m, (\mu m-1)/(m-1)\right)$ affine design, and let $\underline{S}$ be the GD design constructed in Lemma 4.3.1 above. Then $\underline{D}$ constructed as in Section 4.1, is a GD $1\text{-}\left(4\mu m^2 s^2, \mu m(ms(2s-1)+2hs), (\mu m^2-1)(ms(2s-1)+2hs)/(m-1)\right)$ design admitting a strong tactical division with $4s^2$ point classes and $4(\mu m^2-1)s^2/(m-1)$ block classes, of μm^2 points and m blocks each, respectively. The connection number of a point of the i^{th} class and a point of the j^{th} class =

$$(\mu m^2-1)(ms(s-1)+2hs)/(m-1)+\delta_{ij}(4s^2(\mu m-1)(h-\lambda)/(m-1)+s^2(\mu m^2-1)$$

$$(m-4(h-\lambda))/(m-1)).$$

$\underline{D}$ is a 2-design if and only if $\mu = 1$ and $\underline{F}$ is a $2\text{-}(4\lambda+3, 2\lambda+1, \lambda)$ Hadamard design or its complement.

Proof $\underline{D}$ is a GD design with appropriate parameters from Theorems 4.1.1 and 4.1.3 and Lemmas 4.1.2 and 4.1.4(iii). $\underline{D}$ admits a strong tactical division by Lemma 4.1.5 and Theorem 4.1.6. By the above, $\underline{D}$ is a 2-design if and only if

$$4(\mu m-1)s^2(h-\lambda)/(m-1)+(\mu m^2-1)s^2\left[m-4(h-\lambda)\right]/(m-1) = 0.$$

Rearranging and cancelling through by $ms^2/(m-1)$ we obtain: $4(h-\lambda)\mu(m-1)-(\mu m^2-1) = 0$, i.e. $\mu\big(4(h-\lambda)(m-1)-m^2\big)+1 = 0$.

But m,h,λ,μ are positive integers and so this can occur if and only if $\mu = 1$ and $4(h-\lambda)(m-1) = (m^2-1)$, i.e. $\mu = 1$ and $h-\lambda = (m+1)/4$. That is $\underline{M}$ is a symmetric Hadamard design (or its complement) and $\underline{A}$ is an affine plane. ⌑

Theorem 4.3.3 Suppose $m \equiv 3 \pmod 4$ is a prime power and s is such that a $S(s)$ exists. Then there exists a strongly divisible $2-\big(4m^2s^2, ms(2ms-1), (m+1)s(ms-1)\big)$ design $\underline{D}$, with $4s^2$ point classes and $4(m+1)s^2$ block classes of m^2 points and m blocks each, respectively. $\underline{D}$ has intersection numbers:

$$ms(ms-s-1),\ ms(ms-1),\ ms(ms-1) + s^2.$$

Proof Since $m \equiv 3 \pmod 4$ is a prime power, an affine plane $\underline{A}$ of order m exists, i.e. $\underline{A}$ is a $2-(m^2,m,1)$ design. Also by Paley, [34], there exists a Hadamard $2-\big(m,(m-1)/2,(m-3)/4\big)$ design, $\underline{F}$. Let $\underline{E}$ be a $S(s)^c$, then $\underline{S}$ constructed from $\underline{E}$ and $\underline{F}$ using Lemma 4.3.1, is a GD $1-\big(4ms^2, s(2ms-1), s(2ms-1)\big)$ design with connection and intersection numbers $\lambda'_{ij} = \rho'_{ij} = s(ms-1) - s^2\delta_{ij}$. Using Theorem 4.3.2, $\underline{D}$ constructed using $\underline{S}$ and $\underline{A}$ is the required design, and, by Theorem 4.1.7, the intersection numbers of $\underline{D}$ have the above values. ⌑

Remarks Given any strongly divisible 1-design with the parameters of the $\underline{S}$ of Theorem 4.3.3 above, we can construct a strongly divisible 2-design; i.e. it is not necessary for the construction above that $\underline{S}$ belongs to the class of designs constructed in Lemma 4.3.1.

For instance John and Turner, [24], have constructed GD 1-(16,7,7) and 1-(20,9,9) designs with parameters identical to those of the $\underline{S}$ of Theorem 4.3.3 when $s = 1$ and $m = 4$ or 5 respectively. These designs may be used in conjunction with affine planes of orders 4 and 5 to obtain strongly divisible 2-(64,28,15) and 2-(100,45,24) designs.

Since there exist infinitely many primes $\equiv 3 \pmod 4$, and infinitely many S(s) designs exist (see Section 4.2), Theorem 4.3.3 gives an infinite family of strongly divisible 2-designs.

4.4 An Infinite Family of Symmetric 2-designs

Using the construction method of Section 4.1 we now construct a family of strongly divisible 1-designs and 2-designs, and then show that the 2-designs are the residual designs of an infinite family of symmetric 2-designs. This construction may be regarded as a generalisation of the construction of [4], Chapter 4, Section 1, and [9].

It has recently been brought to the attention of the author that, using a different construction method, a family of symmetric designs with the same parameters has been obtained by Rajkundlia, [35].

Suppose $q>2$ is a prime power, and let $h>1$ be any integer. Then put $\underline{A}' = A_{h-1}(h,q)$, the design consisting of the points and hyperplanes of h-dimensional affine geometry over GF(q); (see, for instance, [19]). Choose some point P of $\underline{A}'$, and let $\underline{S} = (\underline{A}')^P$.

<u>Lemma 4.4.1</u> $\underline{S}$ is a symmetric GD $1\text{-}(q^h-1, q^{h-1}, q^{h-1})$ design, and $\underline{S}$ admits a strong tactical division $T(\underline{S})$ with d point and block classes, $d = (q^h-1)/(q-1)$, of $q-1$ points and blocks each. $\underline{S}$ has connection and intersection numbers λ'_{ij} and ρ'_{ij} where $\lambda'_{ij}=\rho'_{ij}=(1-\delta_{ij})q^{h-2}$, $(1\leq i,j\leq d)$. No block of $\underline{S}$ contains all the points of a point class of $\underline{S}$.

<u>Proof</u> From Dembowski, [19], we see that $\underline{A}'$ is an affine $2\text{-}\{q^h, q^{h-1}, (q^{h-1}-1)/(q-1)\}$ design. By Result 1.3.5(iii), $\underline{S} = (\underline{A}')^P$ is a symmetric $1\text{-}(q^h-1, q^{h-1}, q^{h-1})$ design. Define the line of $\underline{A}'$ through two points X and Y to be the intersection of all blocks through X and Y. It is well-known (see, for instance, [19]) that every line of $\underline{A}'$ contains q points. Hence there are d lines, $\ell_1,\ldots,\ell_d$, say, of $\underline{A}'$ containing P; also $\underline{A}'$ has d parallel classes $\underline{A}'_1,\ldots,\underline{A}'_d$ of q blocks each.

Now put $\underline{S}_i = \ell_i-\{P\}$ and $\underline{C}_i = \underline{A}'_i - \{x_i\}$ for every i $(1\leq i\leq d)$, where x_i is the block of $\underline{A}'_i$ which contains P. $\underline{S}_1,\ldots,\underline{S}_d$ and $\underline{C}_1,\ldots,\underline{C}_d$ partition the points and blocks of $\underline{S}$ into d classes of q-1 points and blocks each, and we call this partition $T(\underline{S})$. We now show $T(\underline{S})$ is a tactical decomposition.

Choose a point class $\underline{S}_i$ and a block class $\underline{C}_j$. If $Q \in \underline{S}_i$, then since $\underline{A}'_j$ is a parallel class of $\underline{A}'$, Q is incident with a unique block y, say, of $\underline{A}'_j$. By definition of a line either $\ell_i \cap x_j = \{P\}$ or $\ell_i \subset x_j$. Hence either $\ell_i \cap x_j = \{P\}$, and so $y \neq x_j$, i.e. Q is incident with precisely one block of $\underline{C}_j$, or $\ell_i \subset x_j$ and $y = x_j$, i.e. Q is incident with no blocks of $\underline{C}_j$.

If $z \in \underline{C}_j$ and $\ell_i \cap x_j = \{P\}$ (i.e. $\ell_i \nparallel x_j$), then $\ell_i \cap z \neq \phi$ (since every line meets every non parallel block, and $\ell_i \nparallel z$ since $\ell_i \nparallel x_j$ and $x_j \| z$), and so $\ell_i \cap z = \{R\}$, for some $R \neq P$. If $\ell_i \subset x_j$, then $\ell_i \| x_j, x_j \| z$ and also $\ell_i \subset x_j, x_j \neq z$. So $\ell_i \cap z = \phi$, and hence $T(\underline{S})$ is a tactical decomposition.

Pick any two blocks of $\underline{S}$, $y \in \underline{C}_i$ and $z \in \underline{C}_j$, say. Then if $i=j$, $y,z \in \underline{C}_i \subset \underline{A}'_i$, i.e. y,z are two parallel blocks and so $y \cap z = \phi$. If $i \neq j$, then y,z are two non-parallel blocks of $\underline{A}'$, and so they intersect in q^{h-2} points of $\underline{A}'$; y,z are blocks of $\underline{S}$ and so $P \notin y,z$, i.e. y,z intersect in q^{h-2} points of $\underline{S}$.

So $T(\underline{S})$ forms a tactical division of $\underline{S}^*$, and $\underline{S}$ has intersection numbers $\rho'_{ij} = (1-\delta_{ij})q^{h-2}$, $(1 \leq i,j \leq d)$. Hence, by Theorem 3.3.1 $T(\underline{S})$ is also a tactical division of $\underline{S}$, and $\underline{S}$ has connection numbers $\lambda'_{ij}=\rho'_{ij}$, $(1\leq i,j\leq d)$.

Finally, by above, every block of $\underline{S}$ contains 0 or 1 points from every point class of $T(\underline{S})$, and, since $q>2$, no block of $\underline{S}$ contains all of the $q-1$ points from a point class of $T(\underline{S})$.

Remark To construct $\underline{S}$ with the above properties we needed only that $\underline{A}'$ was an Affine $2-\{\mu m^2,\mu m,(\mu m-1)/(m-1)\}$ design with constant line size m, and so we could replace $A_1(2,q)$ by an arbitrary affine plane of order q.

Theorem 4.4.2 If there exists an affine $2-\{\mu(q-1)^2,\mu(q-1),(\mu(q-1)-1)/(q-2)\}$ design $\underline{A}$, and $q>2$ is a prime power, then for every $h\geq 2$ there exists a GD

$$1-\{\mu(q-1)(q^h-1),\mu q^{h-1}(q-1),q^{h-1}\{\mu(q-1)^2-1\}/(q-2)\} \text{ design } \underline{D}$$

admitting a strong tactical division $T(\underline{D})$ with d point classes each of $\mu(q-1)^2$ points and c block classes $\{c=\{\mu(q-1)^2-1\}(q^h-1)/(q-1)(q-2)\}$ each of $q-1$ blocks. The classes of $T(\underline{D})$ may be labelled so that the connection and intersection numbers are

$$\lambda_{ij}=\{\mu(q-1)^2-1\}q^{h-2}/(q-2)+\delta_{ij}\, q^{h-2}(\mu-1)(q-1)/(q-2),\ (1\leq i,j\leq d),$$

$$\rho=\rho_{ii}=0\quad(1\leq i\leq c),\text{ and } \rho_{ij} \text{ where } \rho_{ij}=\mu q^{h-2}(q-1),\ i\not\equiv j \pmod d\ ;$$

$$\rho_{ij}=\mu q^{h-1},\quad i\equiv j \pmod d\ ,\quad \text{for every } i,j,\ 1\leq i,j\leq c, i\neq j.$$

Proof Since $q>2$ is a prime power let $\underline{S}$ be as in Lemma 4.4.1 above. Then construct $\underline{D}$ using $\underline{S}$ and $\underline{A}$ as in Section 4.1, and

Theorems 4.1.1, 4.1.3, 4.1.6, 4.1.7 and Lemmas 4.1.2, 4.1.5 yield the result. ¤

Corollary 4.4.3 If there exists an affine plane of order $q-1$, and $q>2$ is a prime power, then for every $h\geq 2$ there exists a 2-$\left((q-1)(q^h-1),q^{h-1}(q-1),q^{h-1}\right)$ design $\underline{D}$, admitting a strong tactical decomposition $T(\underline{D})$ with d point classes each of $(q-1)^2$ points and c block classes each of $q-1$ blocks; $\left(c=q(q^h-1)/(q-1)\right)$. $\underline{D}$ has intersection numbers : 0, $q^{h-2}(q-1)$ and q^{h-1}.

Proof $\underline{D}$ of Theorem 4.4.2 is a 2-design if and only if $0 = q^{h-2}(\mu-1)(q-1)/(q-2)$, i.e. if and only if $\underline{A}$ is an affine plane, (since $q>2$). ¤

We now show how to embed the 2-designs of Corollary 4.4.3 into symmetric 2-designs. We first require :

Result 4.4.4 (Beker and Haemers, [7], Corollary 6.3)

A Quasi-residual 2-$\left((k-1)(k-\lambda)/\lambda,k-\lambda,\lambda\right)$ design $\underline{D}$ with three intersection numbers : $0,\lambda(k-\lambda)/k$ and k/m is embeddable in a symmetric 2-(v,k,λ) design if and only if there exists a strongly resolvable 2-$\left(k,\lambda,(\lambda-1)/m\right)$ design $\bar{\underline{D}}$.

We now have :

Theorem 4.4.5 If there exists an affine plane of order $q-1$, and $q>2$ is a prime power, then for every $h\geq 2$ there exists a symmetric 2-$(q^{h+1}-q+1,q^h,q^{h-1})$ design.

Proof By Result 4.4.4 the Quasi-Residual Designs of Corollary 4.4.3 are embeddable in symmetric 2-designs of appropriate parameters if and only if there exists strongly resolvable $2\text{-}(q^h, q^{h-1}, (q^{h-1}-1)/(q-1))$ designs $\underline{\bar{D}}$. But such designs always exist, namely $\underline{\bar{D}} = \underline{A}_{h-1}(h,q)$, and the theorem follows. ¤

Remark Hence, since h may be chosen arbitrarily, we have shown that whenever q and q-1 are prime powers, there exists an infinite family of symmetric 2-designs with the above parameters.

4.5 On a Generalisation of a Construction of Kimberley

It is clear that the construction method of Section 4.1 can be modified and generalised, and below we give an example of how a modified form of the construction can be used to obtain the affine designs of Kimberley, [31].

Suppose $\underline{A}$ is a $2\text{-}(v,k,\lambda)$ structure admitting a parallelism with m blocks in each class, (and hence $v=mk, b=mr$). Let $\underline{S}$ be a $1\text{-}(v',k',r')$ structure admitting a P Division $\underline{S}_1,\ldots,\underline{S}_d$ with $|S_i| = m$ for every i, (and hence $v' = md$). Suppose also that $k' = m$.

Construct $\underline{D}$ from $\underline{A}$ and $\underline{S}$ as in Section 4.1. From Theorems 4.1.1 and 4.1.3 we have :

<u>Lemma 4.5.1</u> $\underline{D}$ is a 1-(mkd,mk,rr') structure admitting a P Division $\underline{P}_1,\ldots,\underline{P}_d$ and $\underline{D}$ has connection numbers

$$r\lambda'_{ij} + \lambda(r'-\lambda'_{ii})\delta_{ij}, \quad (1\leq i,j\leq d).$$

Now construct $\bar{\underline{D}}$ from $\underline{D}$ as follows. Let $\bar{\underline{D}}$ have the same point set as $\underline{D}$ and have as blocks the blocks of $\underline{D}$, (with the same incidence as in $\underline{D}$), together with d further blocks $x_1,\ldots,x_d$ where x_i is incident with the mk points of $\underline{P}_i$. We immediately have (using Lemma 4.5.1) :

<u>Theorem 4.5.2</u> $\bar{\underline{D}}$ is a 1-(mkd,mk,rr'+1) structure admitting a P Division $\underline{P}_1,\ldots,\underline{P}_d$, and $\bar{\underline{D}}$ has connection numbers

$$r\lambda'_{ij} + \delta_{ij}\big(\lambda(r'-\lambda'_{ii}) + 1\big), \quad 1\leq i,j\leq d.$$

We also have :

<u>Lemma 4.5.3</u> $\bar{\underline{D}}$ is a design if the conditions of Lemma 4.1.2 are satisfied.

<u>Proof</u> Clearly no two points of $\bar{\underline{D}}$ are incident with the same set of blocks of $\bar{\underline{D}}$ since $\underline{D}$ is a design, and hence we need only observe that since no blocks of $\underline{S}$ contains all the points of a point class of $\underline{S}$, no block of $\underline{D}$ contains all the points of $\underline{P}_i$ for any i, and so $\bar{\underline{D}}$ is a design. ◻

Lemma 4.5.4 $\underline{\bar{D}}$ is a 2-structure if and only if $\underline{S}_1,\ldots,\underline{S}_d$ is a group division of $\underline{S}$ satisfying

$$r\lambda'_1+\lambda(r-\lambda'_1)+1=r\lambda'_2 \text{ where } \lambda'_1 = \lambda'_{ii} \text{ and } \lambda'_2 = \lambda'_{ij} \text{ for}$$

every i,j, $(1\leq i,j\leq d, i\neq j)$.

Proof Immediate from Theorem 4.5.2. ⌑

Lemma 4.5.5 Suppose $\underline{\bar{D}}$ is a 2-structure and $\underline{S}$ admits a tactical division $T(\underline{S})$ with point classes $\underline{S}_1,\ldots,\underline{S}_d$ and block classes $\underline{C}_1,\ldots,\underline{C}_c$ such that the number of blocks of $\underline{C}_j$ incident with a point of $\underline{S}_i$ depends only on the block class $\underline{C}_j$ (i.e. using the notation of Lemma 4.1.5, for every j $(1\leq j\leq c)$ there exists a γ_j such that $\gamma_{ij}=\gamma_j$ for every i $(1\leq i\leq d)$. Then $\underline{\bar{D}}$ admits a resolution $R(\underline{\bar{D}})$ with $rc+1$ block classes.

Proof Use $T(\underline{S})$ to obtain a tactical decomposition $T(\underline{D})$ of $\underline{D}$ as in Lemma 4.1.5, and, as in the proof of Lemma 4.1.5 any point of $\underline{D}$ is incident with precisely γ_j blocks of $\underline{B}_{\ell j}$ $(1\leq \ell\leq r, 1\leq j\leq c)$. Now let $R(\underline{\bar{D}})$ have block classes the block classes of $T(\underline{D})$ together with one extra block class consisting of the added blocks. $R(\underline{\bar{D}})$ is clearly a resolution of $\underline{\bar{D}}$ with $rc+1$ block classes and the Lemma follows. ⌑

<u>Remark</u> If $T(\underline{S})$ is strong and $\underline{S}$ is a design, then by Theorems 2.2.1 and 2.4.4 the condition $\gamma_{ij} = \gamma_j$ in the Lemma above is equivalent to assuming that the block classes of $T(\underline{S})$ form a SRP Division of $\underline{S}^*$. But if $T(\underline{S})$ is strong, then, by Result 1.5.2, the intersection number of any pair of blocks from the same block class of $T(\underline{S})$ is a constant ("$k-r+\lambda$") and so, by Theorem 2.3.5, the condition $\gamma_{ij} = \gamma_j$ is equivalent to assuming that the block classes of $T(\underline{S})$ form a semiregular group division of $\underline{S}^*$, and hence is equivalent to assuming that both $\underline{S}$ and $\underline{S}^*$ are SRGD; (using Theorem 2.4.5).

We now suppose that $\underline{A}$, $\underline{S}$ and $\underline{\bar{D}}$ are as in Lemma 4.5.5, and we have :

<u>Lemma 4.5.6</u> $R(\underline{\bar{D}})$ is strong if and only if $b'r+d = mkd+rc$.

<u>Proof</u> $R(\underline{\bar{D}})$ is strong if and only if "$b+1 = v+c$", i.e. if and only if $(b'r+d)+1 = mkd+(rc+1)$. ¤

<u>Theorem 4.5.7</u> Any two of the following imply the third :

(i) $\underline{A}$ is affine;

(ii) $T(\underline{S})$ is strong;

(iii) $R(\underline{\bar{D}})$ is strong.

<u>Proof</u> The proof is identical to the proof of Theorem 4.1.6. ¤

We now obtain the construction method of Kimberley, [31], using the above process.

<u>Theorem 4.5.8</u> If there exists an affine $2\text{-}\left(\mu m^2, \mu m, (\mu m-1)/(m-1)\right)$ design and an affine plane of order m, then there exists an affine $2\text{-}\left(\mu m^3, \mu m^2, (\mu m^2-1)/(m-1)\right)$ design.

<u>Proof</u> Let $\underline{A}$ be the affine $2\text{-}\left(\mu m^2, \mu m, (\mu m-1)/(m-1)\right)$ design, and $\underline{S}'$ be the affine plane of order m. Suppose the parallel classes of $\underline{S}'$ are $\underline{U}_1, \ldots, \underline{U}_{m+1}$ and then let $\underline{S}$ be the incidence structure whose points are the points of $\underline{S}'$, and whose blocks are the blocks of $\underline{U}_1, \ldots, \underline{U}_m$ with incidence as in $\underline{S}'$. Clearly $\underline{S}$ is a $1\text{-}(m^2,m,m)$ design. Define $T(\underline{S})$ to have point classes the point sets of the blocks of $\underline{U}_{m+1}$ and block classes $\underline{U}_1, \ldots, \underline{U}_m$.

Then $T(\underline{S})$ has m point and block classes of m points and blocks each. Every point of $\underline{S}$ is incident with precisely one block of any given block class and any two points of $\underline{S}$ are on 0 or 1 common blocks depending only on whether they are from the same or different point classes respectively. Dually every block of $\underline{S}$ is incident with precisely one point of any given point class.

Hence $\underline{S}$ is a symmetric SRGD design (by Result 1.4.5) admitting a strong tactical division with point classes the classes of the group division. By above, every block of $\underline{S}$ contains precisely 1 point from every point class of $T(\underline{S})$, and so no block of $\underline{S}$ contains all of the m points of a point class of $\underline{S}$.

Hence constructing $\underline{\bar{D}}$ from $\underline{A}$ and $\underline{S}$ as above, $\underline{\bar{D}}$ is a $2\text{-}(\mu m^3, \mu m^2, (\mu m^2-1)/(m-1))$ design admitting a strong resolution $R(\underline{\bar{D}})$, (using Theorems 4.5.2, 4.5.7 and Lemmas 4.5.3, 4.5.4 and 4.5.5). By Result 1.5.3 $\underline{\bar{D}}$ has intersection numbers 0 and m and so $R(\underline{\bar{D}})$ is a parallelism. Thus $\underline{\bar{D}}$ is affine by Result 1.5.5. ¤

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