Orientable and negative orientable sequences

Chris J. Mitchell and Peter R. Wild
Information Security Group, Royal Holloway, University of London
me@chrismitchell.net; peterrwild@gmail.com

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Abstract

Analogously to de Bruijn sequences, orientable sequences have application in automatic position-location applications and, until recently, studies of these sequences focused on the binary case. In recent work by Alhakim et al., a range of methods of construction were described for orientable sequences over arbitrary finite alphabets; some of these methods involve using negative orientable sequences as a building block. In this paper we describe three techniques for generating such negative orientable sequences, as well as upper bounds on their period. We then go on to show how these negative orientable sequences can be used to generate orientable sequences for every non-binary alphabet size and for every tuple length. In doing so we use two closely related approaches described by Alhakim et al. The periods of both negative orientable and orientable sequences that we construct are of the same order of magnitude as the upper bounds.

1 Introduction

Orientable sequences are periodic sequences with elements drawn from a finite alphabet with the property that any subsequence of n consecutive elements (an n-tuple) occurs at most once *in either direction*; they were introduced in 1992 [3, 4, 7]. For example, the sequence over $\{0,1,2\}$ with period [012] is orientable as the 2-tuples it contains are $(0,1),(1,2),(2,0)$ in the forward direction and $(0,2),(2,1),(1,0)$ in the reverse direction. They are of interest due to their application in certain position-resolution scenarios. For the binary case, a construction and an upper bound on the period were established by Dai et al. [7], and further constructions were established by Mitchell and Wild [14] and Gabrić and Sawada [9, 10]. A bound on the period and methods of construction for q-ary alphabet sequences (for arbitrary q) were given by Alhakim et al. [2].

In this paper we use two of the methods proposed in [2] to construct q-ary orientable sequences for arbitrary n and q. This provides orientable sequences with period of the same order of magnitude as the upper bound. Both these methods involve the use of q-ary negative orientable sequences as a building block. Negative orientable sequences, first introduced in [2], are analogous to orientable sequences except that any n-tuple occurs at most once in either a period of the sequence or a period of the negative reversed sequence. Interpreting the alphabet of the example above as residues modulo 3, so that -0=0, -1=2 and -2=1, the example above is not negative orientable as the negatives of the reverse 2-tuples are $(0,1),(1,2),(2,0)$, equal to the forward 2-tuples. The sequence, over the same alphabet, with period [011] is negative orientable (but not orientable) as the forward 2-tuples are $(0,1),(1,1),(1,0)$ and the negatives of the reverse 2-tuples are $(2,0),(2,2),(0,2)$.

We first demonstrate methods for generating negative orientable sequences for every $q > 2$ and n. All these sequences can in turn be used to generate orientable sequences via the inverse Lempel Homomorphism, following [2]. This parallels recent work by Gabrić and Sawada [11] who also constructed q-ary orientable sequences for arbitrary n and q, albeit using a different approach.

The main motivation in studying negative orientable sequences has been to help construct new examples of orientable sequences. However, they may have applications of their own, and potentially merit further study.

The remainder of this paper is structured as follows. In the rest of Section 1, basic concepts are introduced. Section 2 describes how maximal period orientable sequences can be constructed for every alphabet size $q > 2$ when the window size is 2. Section 3 establishes an upper bound on the period of negative orientable sequences and gives three methods of construction for such sequences. In Sections 4 and 5, we show how the negative orientable sequences constructed in Section 3 can be used to construct new orientable sequences. Finally, Section 6 concludes the paper, and suggests directions for future work.

1.1 Basic terminology

We first establish some simple notation, following [2]. For mathematical convenience we consider the elements of a sequence to be elements of $\mathbb{Z}_q$ for an arbitrary integer $q > 1$.

For a sequence $S = (s_i)$ we write $\mathbf{s}_n(i) = (s_i, s_{i+1}, \ldots, s_{i+n-1})$. Since we are interested in tuples occurring either forwards or backwards in a sequence we also introduce the notion of a reversed tuple, so that if $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is a q-ary n-tuple (a string of symbols of length n) then $\mathbf{u}^R = (u_{n-1}, u_{n-2}, \ldots, u_0)$ is its *reverse*. The *negative* of a q-ary n-tuple

$\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is the n-tuple $-\mathbf{u} = (-u_0, -u_1, \ldots, -u_{n-1})$.

We can then give the following.

Definition 1.1 ([2]) *A q-ary n-*window sequence *$S = (s_i)$ is a periodic sequence of elements from $\mathbb{Z}_q$ ($q > 1$, $n > 1$) with the property that no n-tuple appears more than once in a period of the sequence, i.e. with the property that if $\mathbf{s}_n(i) = \mathbf{s}_n(j)$ for some i, j, then $i \equiv j \pmod m$ where m is the period of the sequence.*

Definition 1.2 ([2]) *A q-ary n-window sequence $S = (s_i)$ is said to be an* orientable sequence of order *n (an $\mathcal{OS}_q(n)$) if $s_n(i) \neq s_n(j)^R$, for any i, j.*

A range of examples of orientable sequences can be found in Gabrić and Sawada [9]. We also need two related definitions.

Definition 1.3 ([2]) *A q-ary n-window sequence $S = (s_i)$ is said to be a* negative orientable sequence of order *n (a $\mathcal{NOS}_q(n)$) if $\mathbf{s}_n(i) \neq -\mathbf{s}_n(j)^R$, for any i, j.*

Examples of negative orientable sequences are given in Section 3.5. As discussed in Alhalkim et al. [2], it turns out that negative orientable sequences can be used to construct orientable sequences. Observe that a sequence is orientable if and only if it is negative orientable for the case $q = 2$. Also note that if $S = (s_i)$ is orientable or negative orientable then so is its negative $(-s_i)$.

1.2 Related work

This paper builds on the work of Alhakim et al. [2], in which recursive methods of construction for non-binary orientable sequences are described. In particular, methods for developing sequences suitable for use as 'starter sequences' in the recursive constructions are developed.

The work described here is complementary to parallel recent work by Gabrić and Sawada [11]. They show how to join a set of q-ary cycles to obtain q-ary orientable sequences with asymptotically maximal periods. Their method is suited to rapid computation, enabling construction of q-ary orientable sequences of order n in $O(n)$ time per symbol and $O(n)$ space.

2 Constructing maximal orientable sequences for $n = 2$

We start by showing that, unlike the binary case, orientable sequences for $n = 2$ with period meeting the bound of [2] can easily be constructed for every $q > 2$. Analogous results have recently been obtained by Gabrić and Sawada [11].

The bound of Theorem 4.11 of [2] is as follows: Suppose that $S = (s_i)$ is an $\mathcal{OS}_q(n)$ ($q \geq 2$, $n \geq 2$). Then the period of S is at most

$$\begin{aligned} (q^n - q^{\lceil n/2 \rceil} - q^{\lceil (n-1)/2 \rceil} + q)/2 & \quad \text{if } q \text{ is odd,} \\ (q^n - q^{\lceil n/2 \rceil} - q)/2 & \quad \text{if } q \text{ is even.} \end{aligned}$$

Further, if q is odd and $n \geq 6$ then the period of S is at most

$$\begin{aligned} (q^n - 2q^{n/2} - q^{(n-2)/2} + 2q)/2 & \quad \text{if } n \text{ is even,} \\ (q^n - q^{(n+1)/2} - 2q^{(n-1)/2} + q + q^2)/2 & \quad \text{if } n \text{ is odd.} \end{aligned}$$

We note that all these bounds are asymptotically of the same order as $q^n/2$.

Definition 2.1 *Let S be an $OS_q(n)$. If S has period meeting the bound of Theorem 4.11 of Alhakim et al. [2], as given immediately above, then S is said to be* maximal.

Lemma 2.2 *There exists a maximal $OS_q(2)$ for all $q \geq 3$.*

Proof First observe that the ring sequence corresponding to an $OS_q(2)$ of period p corresponds to the list of vertices visited in a circuit (i.e. a closed path in the graph) of length p in the complete graph K_q with its q vertices labelled $0, 1, \ldots, q-1$.

If q is odd, then, by [2, Theorem 4.11], a maximal $OS_q(2)$ S has period $q(q-1)/2$, i.e. it corresponds to an Eulerian circuit in the complete graph on q vertices, K_q (since K_q has $q(q-1)/2$ edges). Every vertex of K_q has degree $q-1$, which is even since q is odd, and hence such a circuit always exists by Euler's Theorem (see, for example, Corollary 6.1 of Gibbons [12]), and the result follows.

If q is even, then, by [2, Theorem 4.11], a maximal $OS_q(2)$ S has period $q(q-2)/2$. Let K_q^* be the complete graph with an arbitrary 1-factor removed; for example, if the vertices are labelled $0, 1, \ldots, q-1$, we could remove the edges $(2i, 2i+1)$ for every i, $0 \leq i < q/2$. It is simple to observe that K_q^* has $q(q-2)/2$ edges and every vertex has degree $q-2$, which is even. The desired result again follows from Euler's Theorem. ∎

Remark 2.3 *There are simple algorithms for finding Eulerian circuits — see for example Gibbons [12, Figure 6.5]. Moreover, a direct construction of an* $OS_q(2)$ *for the case where* $q > 2$ *is a prime is presented in [2, Construction 5.3].*

3 Negative orientable sequences

We next establish some fundamental results on negative orientable sequences (see Definition 1.3), given their importance in constructing orientable sequences using the methods of [2]. We further describe three methods of construction for such sequences. Here and throughout the remainder of the paper we only consider the case $q > 2$ for two main reasons. As observed in Section 1, when $q = 2$, negative orientable sequences are the same as orientable sequences. Further, the methods of construction we present in this section do not work in the case $q = 2$.

3.1 A bound on the period

We start by giving a bound on the period of negative orientable sequences. To establish this bound we need to introduce the following terminology to distinguish a special class of n-tuples. A q-ary n-tuple $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is said to be *negasymmetric* if $u_i = -u_{n-1-i}$ for every i $(0 \leq i \leq n-1)$, i.e. if $\mathbf{u} = -\mathbf{u}^R$. Clearly an $\mathcal{NOS}_q(n)$ cannot contain any negasymmetric n-tuples. This motivates the following.

Lemma 3.1 *For* $n \geq 2$*, the number of* q*-ary negasymmetric* n*-tuples is*

$$\begin{array}{rl} q^{n/2} & \text{if } n \text{ is even} \\ q^{(n-1)/2} & \text{if } n \text{ is odd and } q \text{ is odd} \\ 2q^{(n-1)/2} & \text{if } n \text{ is odd and } q \text{ is even.} \end{array}$$

Proof The n even case is immediate. For n odd ($n = 2m+1$ say), an n-tuple $(u_0, u_1, \ldots, u_{m-1}, u_m, -u_{m-1}, \ldots, -u_1, -u_0)$ is negasymmetric if and only if $u_m = -u_m$. If q is odd this implies $u_m = 0$, and if q is even then this implies u_m is either 0 or $q/2$. The result follows. ∎

This gives the following simple bound.

Corollary 3.2 *For* $n \geq 2$*, the period of an* $\mathcal{NOS}_q(n)$ *is at most*

$$\begin{array}{rl} (q^n - q^{n/2})/2 & \text{if } n \text{ is even} \\ (q^n - q^{(n-1)/2})/2 & \text{if } n \text{ is odd and } q \text{ is odd} \\ (q^n - 2q^{(n-1)/2})/2 & \text{if } n \text{ is odd and } q \text{ is even.} \end{array}$$

Proof There are trivially q^n q-ary n-tuples, of which only those that are non-negasymmetric can occur in a $\mathcal{NOS}_q(n)$. If $\mathbf{u}$ is such an n-tuple, then at most one of $\mathbf{u}$ and $-\mathbf{u}^R$ can occur in an $\mathcal{NOS}_q(n)$. The result follows. $\blacksquare$

As we next show, it is possible to improve on this simple bound. We first need the following.

Definition 3.3 *An n-tuple $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$, $u_i \in \mathbb{Z}_q$ $(0 \le i \le n-1)$, is said to be* m-negasymmetric *for some $m \le n$ if and only if the m-tuple $(u_0, u_1, \ldots, u_{m-1})$ is negasymmetric.*

Clearly an n-tuple is n-negasymmetric if and only if is it is negasymmetric. We also need the notion of uniformity.

Definition 3.4 *An n-tuple $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$, $u_i \in \mathbb{Z}_q$ $(0 \le i \le n-1)$, is* c-uniform *for some $c \in \mathbb{Z}_q$ if and only if $u_i = c$ for every i $(0 \le i \le n-1)$.*

We can then state the following elementary results.

Lemma 3.5 *If $n \ge 2$ and $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is a q-ary n-tuple that is both negasymmetric and $(n-1)$-negasymmetric, then $\mathbf{u}$ is c-uniform where c is either 0 or $q/2$ (the latter only applying if q is even).*

Proof Choose any i, $0 \le i \le n-2$. Then, by negasymmetry we know that $u_{i+1} = -u_{n-2-i}$ (observing that $i \le n-2$). Also, by $(n-1)$-negasymmetry we know that $u_i = -u_{n-2-i}$, and hence we have $u_i = u_{i+1}$. The result follows. $\blacksquare$

Definition 3.6 *Let $N^*_q(n)$ be the set of all non-negasymmetric q-ary n-tuples.*

The size of $N^*_q(n)$ follows from Lemma 3.1. As a first step towards establishing our bound we need to define a special set of n-tuples, as follows.

Definition 3.7 ([2]) *Suppose $n \ge 2$, and that $\mathbf{v} = (v_0, v_1, \ldots, v_{n-2})$ is a q-ary $(n-1)$-tuple. Then let $L(\mathbf{v})$ be the following set of q-ary n-tuples:*

$$L(\mathbf{v}) = \{\mathbf{u} = (u_0, u_1, \ldots, u_{n-1}) : \ u_i = v_i, \quad 0 \le i \le n-2\}.$$

That is, $L(\mathbf{v})$ is simply the set of n-tuples whose first $n-1$ entries equal $\mathbf{v}$. Clearly, the sets $L(\mathbf{v})$ for all $(n-1)$-tuples $\mathbf{v}$ are disjoint. We are interested in how the sets $L(\mathbf{v})$ intersect with the set of n-tuples occurring in either S or $-S^R$ when S is a $\mathcal{NOS}_q(n)$ and $\mathbf{v}$ is a negasymmetric $(n-1)$-tuple.

Definition 3.8 *Suppose $n \geq 2$, $r \geq 1$, $S = (s_i)$ is a $\mathcal{NOS}_q(n)$, and $\mathbf{v} = (v_0, v_1, \ldots, v_{n-2})$ is a q-ary $(n-1)$-tuple. Then let*

$$L_S^*(\mathbf{v}) = \{\mathbf{u} \in L(\mathbf{v}) : \ \mathbf{u} \text{ appears in } S \text{ or } -S^R\}.$$

It is important to observe that, from the definition of negative orientable, an element of $L_S^*(\mathbf{v})$, as with any n-tuple, can only appear at most once in S or $-S^R$. We now state a result on which our bound is built.

Lemma 3.9 *Suppose $n \geq 2$, $S = (s_i)$ is a $\mathcal{NOS}_q(n)$, and $\mathbf{v} = (v_0, v_1, \ldots, v_{n-2})$ is a q-ary negasymmetric $(n-1)$-tuple. Then $|L_S^*(\mathbf{v})|$ is even.*

Proof Suppose $L_S^*(\mathbf{v})$ is non-empty. Then, by definition, and (without loss of generality) assuming that an element of $L(\mathbf{v})$ occurs in S (as opposed to $-S^R$), we know that $v_i = s_{j+i}$ for some j $(0 \leq i \leq n-2)$. Since $\mathbf{v}$ is negasymmetric, it follows immediately that $v_i = -s_{j+n-2-i}$ $(0 \leq i \leq n-2)$. That is, for each occurrence of an element of $L(\mathbf{v})$ in S, there is an occurrence of a, necessarily distinct, element of $L(\mathbf{v})$ in $-S^R$, and vice versa. The result follows. ∎

If $|L(\mathbf{v})|$ is odd, Lemma 3.9 immediately shows that S and $-S^R$ combined must omit at least one of the n-tuples in $L(\mathbf{v})$. We can now state our main result.

Theorem 3.10 *Suppose that $S = (s_i)$ is a $\mathcal{NOS}_q(n)$ $(q \geq 2,\ n \geq 2)$. Then the period of S is at most*

$$\begin{aligned} (q^n - q^{\lfloor n/2 \rfloor} - q^{\lfloor (n-1)/2 \rfloor} + 1)/2 & \quad \text{if } q \text{ is odd,} \\ (q^n - 2q^{(n-1)/2})/2 - 1 & \quad \text{if } q \text{ is even and } n \text{ is odd,} \\ (q^n - q^{n/2})/2 - 1 & \quad \text{if } q \text{ is even and } n \text{ is even.} \end{aligned}$$

Proof The proof involves considering the set $N_q^*(n)$ of non-negasymmetric q-ary n-tuples, and showing that the tuples in certain disjoint subsets of this set cannot all occur in a $\mathcal{NOS}_q(n)$. We divide our discussion into two cases depending on the parity of q.

- First suppose that q is odd. Suppose also that $\mathbf{v}$ is a negasymmetric non-uniform $(n-1)$-tuple. From Lemma 3.1 there are $q^{\lfloor (n-1)/2 \rfloor}$ negasymmetric $(n-1)$-tuples of which precisely one, namely the all-zero tuple, is uniform — since q is odd; i.e. there are $q^{\lfloor (n-1)/2 \rfloor} - 1$ such $\mathbf{v}$. From Lemma 3.5, none of the q elements of $L(\mathbf{v})$ are negasymmetric, and hence $|L(\mathbf{v}) \cap N_q^*(n)| = q$. Since $|L(\mathbf{v})| = q$ is odd, Lemma 3.9 implies that $L_S^*(\mathbf{v})$ must omit at least one of the n-tuples in $L(\mathbf{v})$. That is, there are $q^{\lfloor (n-1)/2 \rfloor} - 1$ non-negasymmetric n-tuples which cannot appear in S or $-S$. The bound for q odd follows from Corollary 3.2.

- Now suppose q is even. Let $\mathbf{v}$ be a c-uniform $(n-1)$-tuple for $c = 0$ or $c = q/2$. There are clearly two such $(n-1)$-tuples. By Lemma 3.5 precisely one of the elements of $L_S^*(\mathbf{v})$ is negasymmetric, namely the relevant c-uniform n-tuple, and hence $|L(\mathbf{v}) \cap N_q^*(n)| = q - 1$. Now, since q is even, Lemma 3.9 implies that at least one element of $L(\mathbf{v}) \cap N_q^*(n)$ cannot be contained in $L_S^*(\mathbf{v})$. The result for q even follows from Corollary 3.2. ∎

We note that, as in the case of orientable sequences, all these bounds on negative orientable sequences are asymptotically of the same order as $q^n/2$. Any $\mathcal{NOS}_q(n)$ with period meeting the above bound is said to have maximal period.

We conclude by tabulating the values of the bounds of Theorem 3.10 for small q and n.

Table 1: Bounds on the period of a $\mathcal{NOS}_q(n)$ (from Theorem 3.10)

	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$
$q = 2$	0	1	5	11	27	55	119
$q = 3$	3	11	35	113	347	1067	3227
$q = 4$	5	27	119	495	2015	8127	32639
$q = 5$	10	58	298	1538	7738	38938	194938
$q = 6$	14	101	629	3851	23219	139751	839159
$q = 7$	21	165	1173	8355	58629	411429	2881029
$q = 8$	27	247	2015	16319	130815	1048063	8386559

For the case $n = 2$, the bound simplifies to $(q^2 - q)/2$ for q odd, and $(q^2 - q)/2 - 1$ for q even. As we next show, in this case the bound is tight.

3.2 Construction I: Sequences with $n = 2$

The first method of construction we present provides negative orientable sequences of maximal period for every $q > 3$ in the case $n = 2$. The method involves joining certain explicitly defined cycles.

Consider the directed graph G_q with vertex set $\mathbb{Z}_q$, $q \geq 3$, and all directed arcs (x, y), $x, y \in \mathbb{Z}_q$, except for $(x, -x)$, $x \in \mathbb{Z}_q$. Then a $\mathcal{NOS}_q(2)$ S corresponds to a circuit in G_q. Moreover $-S^R$ is also such a circuit and these two circuits are arc-disjoint. We say that a circuit $C = [c_0, c_1, \ldots, c_{m-1}]$ of length m, consisting of arcs (c_i, c_{i+1}), $i = 0, \ldots, m-1$, identifying c_m with c_0, in G_q is *nice* if it is arc-disjoint from $-C^R$. We note that if C' is another nice circuit in G_q that is arc-disjoint from C and $-C^R$ and if it shares a vertex with C then C and C' may be joined to make a nice circuit of length the sum of the lengths of C and C'.

Suppose that q is odd. Consider the following circuits in G_q.

$$\begin{aligned}
C_0 &= \left[0110220\ldots0\frac{q-1}{2}\frac{q-1}{2}\right] \quad \text{of length } 3\frac{q-1}{2}\\
C_1 &= \left[121(-2)131(-3)1\ldots1\frac{q-1}{2}1(-\frac{q-1}{2})\right] \quad \text{of length } 4\frac{q-3}{2}\\
&\vdots\\
C_i &= \left[i(i+1)i-(i+1)i\ldots i\frac{q-1}{2}i-\frac{q-1}{2}\right] \quad \text{of length } 4\frac{q-i-2}{2}\\
&\vdots\\
C_{\frac{q-3}{2}} &= \left[\frac{q-3}{2}\frac{q-1}{2}\frac{q-3}{2}-\frac{q-1}{2}\right] \quad \text{of length } 4.
\end{aligned}$$

These circuits are arc-disjoint since by construction the arcs of C_i involve the vertex i, with the exception of the loops (j,j) in C_0, in such a way that there is no repetition. Moreover they are arc-disjoint to $-C_i^R$, $i=0,\ldots,\frac{q-3}{2}$ since by construction the arcs of $-C_i^R$ involve the vertices $-j$ in such a way that there is no repetition. Thus they are mutually nice circuits and it may be seen that these circuits may be joined to give a $\mathcal{NOS}_q(2)$ C. A simple calculation shows that C has period $\frac{q(q-1)}{2}$ and therefore is maximal by Theorem 3.10.

Suppose that q is even. Consider the following circuits in G_q.

$$\begin{aligned}
C_0 &= \left[0110220\ldots0\frac{q-2}{2}\frac{q-2}{2}\right] \quad \text{of length } 3\frac{q-2}{2}\\
C_1 &= \left[121(-2)131(-3)1\ldots1\frac{q-2}{2}1(-\frac{q-2}{2})1\frac{q}{2}\right] \quad \text{of length } 4\frac{q-4}{2}+2\\
&\vdots\\
C_i &= \left[i(i+1)i-(i+1)i\ldots i\frac{q-2}{2}i-\frac{q-2}{2}i\frac{q}{2}\right] \quad \text{of length } 4\frac{q-2i-2}{2}+2\\
&\vdots\\
C_{\frac{q-4}{2}} &= \left[\frac{q-4}{2}\frac{q-2}{2}\frac{q-4}{2}-\frac{q-2}{2}\frac{q-4}{2}\frac{q}{2}\right] \quad \text{of length } 6\\
C_{\frac{q-2}{2}} &= \left[\frac{q-2}{2}\frac{q}{2}\right] \quad \text{of length } 2.
\end{aligned}$$

By a similar analysis it can be seen that these circuits may be joined to give a $\mathcal{NOS}_q(2)$ C. A straightforward calculation shows that C has period $\frac{q(q-1)}{2} - 1$ and therefore is maximal, again by Theorem 3.10.

These examples establish the following Lemma by construction.

Lemma 3.11 *There exists a maximal $\mathcal{NOS}_q(2)$ for all $q \geq 3$.*

Note 3.12 *Observe that the maximal $\mathcal{NOS}_q(2)$ constructed using the approaches just described all possess the i-uniform 2-tuples (i, i) for $1 \leq i < q/2$.*

3.3 Construction II: A construction for general n

The second method of construction involves the notion of the pseudoweight of an n-tuple, which is equal to the sum of the values in a tuple after changing all the zeros to $q/2$. The construction then builds on the observation that if the pseudoweights of all the tuples in a sequence are less than half the maximum possible value ($nq/2$) then the pseudoweight of all the n-tuples in the negative sequence will be greater than half the maximum possible value.

We first need the following formal definition of pseudoweight.

Definition 3.13 *Suppose $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is an n-tuple of elements of $\mathbb{Z}_q$ $(q > 1)$. Define the function $f : \mathbb{Z}_q \to \mathbb{Q}$ as follows: for any $u \in \mathbb{Z}_q$ treat u as an integer in the range $[0, q-1]$ and set $f(u) = u$ if $u \neq 0$ and $f(u) = q/2$ if $u = 0$. Then the* pseudoweight *of $\mathbf{u}$ is defined to be the sum*

$$w^*(\mathbf{u}) = \sum_{i=0}^{n-1} f(u_i)$$

where the sum is computed in $\mathbb{Q}$.

As a simple example for $q = 3$, the 4-tuple $(0, 1, 1, 2)$ has weight $0+1+1+2 = 4$ and pseudoweight $1.5 + 1 + 1 + 2 = 5.5$, since $f(0) = \frac{3}{2}$.

Theorem 3.14 *Suppose S is a q-ary n-window sequence $(n \geq 2)$ with the property that all the n-tuples appearing in S have pseudoweight less than $nq/2$. Then S is a $\mathcal{NOS}_q(n)$.*

Proof Consider any n-tuple $\mathbf{u}$ occurring in S, and by definition we know that $w^*(\mathbf{u}) < nq/2$. We claim that $w^*(-\mathbf{u}^R) = nq - w^*(\mathbf{u})$. This follows immediately from the definition of f since $f(-u_i) = q - f(u_i)$ for every possible value of u_i.

Hence, since $w^*(\mathbf{u}) < nq/2$, it follows immediately that $w^*(-\mathbf{u}^R) > nq/2$. Thus the n-tuples in $-S^R$ are all distinct from the n-tuples in S. Moreover, the n-tuples in S (and in $-S^R$) are all distinct because S is an n-window sequence. Hence S is a $\mathcal{NOS}_q(n)$. ∎

It remains to establish how to construct the desired sequences S, i.e. sequences in which all the constituent n-tuples have pseudoweight less than $nq/2$. Suppose, for $q > 2$ and $n \geq 2$, G is a directed graph with vertices the q-ary $(n-1)$-tuples with pseudoweight less than $nq/2 - 1$, and with a directed edge connecting vertices $\mathbf{u} = (u_0, u_1, \ldots, u_{n-2})$ and $\mathbf{v} = (v_0, v_1, \ldots, v_{n-2})$ if and only if $u_i = v_{i-1}$, $(1 \leq i \leq n-2)$, and $w^*(\mathbf{u}) + f(v_{n-2}) < nq/2$, where f is as in Definition 3.13. As is conventional, we identify the edge connecting $(u_0, u_1, \ldots, u_{n-2})$ to $(u_1, u_2, \ldots, u_{n-2}, v)$ with the n-tuple $(u_0, u_1, \ldots, u_{n-2}, v)$, and G contains as an edge every q-ary n-tuple with pseudoweight less than $nq/2$.

A directed circuit in G will clearly give us a sequence S with the property that all its n-tuples have pseudoweight less than $nq/2$. Consider any vertex $\mathbf{u} = (u_0, u_1, \ldots, u_{n-2})$. An incoming edge

$$(s, u_0, u_1, \ldots, u_{n-2})$$

must satisfy $s + w^*(\mathbf{u}) < nq/2$. Similarly an outgoing edge

$$(u_0, u_1, \ldots, u_{n-2}, t)$$

must satisfy $t + w^*(\mathbf{u}) < nq/2$. That is, the in-degree of every vertex is the same as its out-degree. Moreover, G is clearly connected, as it is straightforward to construct a 'low pseudoweight' directed path between any pair of vertices — that is, for any vertex $\mathbf{u} = (u_0, u_1, \ldots, u_{n-2})$ there is a path to $(1, 1, \ldots, 1)$ via vertices of the form $(u_i, u_{i+1}, \ldots, u_{n-2}, 1, 1, \ldots, 1)$ and similarly from $(1, 1, \ldots, 1)$ to any vertex $\mathbf{v}$. This means there exists a directed Eulerian Circuit in G (see, for example, [12, Corollary 6.1]), yielding a q-ary n-window sequence S with the property that all the n-tuples appearing in S have weight less than $nq/2$, and with period equal to the number of q-ary n-tuples with pseudoweight less than $nq/2$.

Let $r_{q,n,s}$ $(q \geq 2,\ n \geq 1)$ denote the number of q-ary n-tuples with pseudoweight exactly s, where $r_{q,n,s} = 0$ by definition if $s < n$ or $s > n(q-1)$. Then the above discussion establishes the following result.

Lemma 3.15 *There exists a $\mathcal{NOS}_q(n)$ of period $\frac{q^n - r_{q,n,nq/2}}{2}$ for all $q > 2$ and $n \geq 2$.*

Proof There are trivially q^n q-ary n-tuples, and for every n-tuple $\mathbf{u}$ of pseudoweight less than $nq/2$ there is a corresponding 'pseudonegative' n-tuple $-\mathbf{u}$ of weight greater than $nq/2$. The result follows. ∎

3.4 Enumerating n-tuples with given pseudoweight

Lemma 3.15 means that it is of interest to know the value of $r_{q,n,s}$. If q is even then $r_{q,n,s}$ is only defined for integer values s, and if q is odd it is only defined when s is a multiple of 0.5. We have the following elementary result.

Lemma 3.16 *Suppose $q \geq 2$ and $n \geq 1$.*

(i) If q is even, $r_{q,1,i} = 1$ for $1 \leq i \leq q-1$, $i \neq q/2$, and $r_{q,1,q/2} = 2$.

(ii) If q is odd, $r_{q,1,i} = 1$ for $1 \leq i \leq q-1$, and $r_{q,1,q/2} = 1$.

(iii) If q is even, $\sum_{i=n}^{n(q-1)} r_{q,n,i} = q^n$.

(iv) If q is odd, $\sum_{j=0}^{2n(q-2)} r_{q,n,n+j/2} = q^n$.

(v) If $n > 1$, $r_{q,n,s} = \sum_{i=1}^{q-1} r_{q,n-1,s-i} + r_{q,n-1,s-q/2}$.

Proof Parts (i)–(iv) are trivially true (observing that the 1-tuples (0) and $(q/2)$ both have pseudoweight $q/2$). Part (v) follows immediately by considering the q possible ways of adding a final entry to an $(n-1)$-tuple to make an n-tuple of a given pseudoweight, observing that inserting 0 as the final entry will increase the pseudoweight by $q/2$. ∎

We can also make some simple observations.

Corollary 3.17 *Suppose $q \geq 2$.*

(i) If q is odd and $2 \leq w \leq 2q-2$, then $r_{q,2,q} = q$ and $r_{q,2,w} < q$ if $w \neq q$.

(ii) If q is even and $2 \leq w \leq 2q-2$, then $r_{q,2,q} = q+2$ and $r_{q,2,w} \leq q$ if $w \neq q$.

(iii) If q is odd, $n \geq 3$ and $n \leq w \leq n(q-1)$, then $r_{q,n,w} < q^{n-1}$.

(iv) If q is even, $n \geq 3$ and $n \leq w \leq n(q-1)$, then $r_{q,n,w} \leq q^{n-3}(q^2+2)$.

(v) If $n \geq 1$ and $0 \leq i \leq 2n$ then $r_{3,n,n+i/2} = N_2(i,n)$, where $N_2(i,n)$ is a trinomial coefficient (see Section 3.7 below).

(vi) If $1 \leq n \leq w \leq 3n$, then $r_{4,n,w} = \binom{2n}{w-n}$.

Proof (i) If q is odd, then, by Lemma 3.16(v),

$$r_{q,2,q} = \sum_{i=1}^{q-1} r_{q,1,q-i} + r_{q,1,q-q/2} = (q-1)+1, \text{(by Lemma 3.16(ii))}.$$

If $w \neq q$, then, by the same result, $r_{q,2,w}$ will be the sum of q terms at most $q-1$ of which are non-zero, all equal to 1, i.e. $r_{q,2,w} \leq q-1$.

(ii) If q is even then, by Lemma 3.16(v),

$$r_{q,2,q} = \sum_{i=1}^{q-1} r_{q,1,i} + r_{q,1,q/2} = q + 2, \text{(by Lemma 3.16(i))}.$$

If $w \neq q$, then, by the same result, $r_{q,2,w}$ is equal to the sum of q terms at most $q-1$ of which are non-zero, with at most one equal to 2 and all others no greater than 2 ($r_{q,n-1,w-q/2}$ is at most 1 since $w \neq q$). Hence $r_{q,2,w} \leq (q-2)+2 = q$.

(iii) This follows by induction on n. If $n = 3$ then, by Lemma 3.16(v), $r_{q,3,w}$ is the sum of q terms, all of which are less than q except possibly at most one equal to q — from (i). Hence, since $q > 2$, $r_{q,3,w} \leq (q-1)^2 + q < q^2$, and the claim holds. The induction step is trivial since $r_{q,m+1,w}$ is equal to the sum of q terms $r_{q,m,i}$, each of which is less than q^{m-1}.

(iv) This also follows by induction on n. If $n = 3$ then, by Lemma 3.16(v), $r_{q,3,w}$ is the sum of q terms, all of which are at most q except possibly at most one equal to $q+2$) — from (ii). Hence, $r_{q,3,w} \leq (q-1)q + (q+2) = q^2 + 2$, and the claim holds. The induction step is trivial since $r_{q,m+1,w}$ is equal to the sum of q terms $r_{q,m,i}$, each of which is at most $q^{m-3}(q^2+2)$.

(v) When $q = 3$ the recursion of Lemma 3.16(v) is $r_{3,n,s} = r_{3,n-1,s-2} + r_{3,n-1,s-3/2} + r_{3,n-1,s-1}$ and by Lemma 3.16(ii) we have $r_{3,1,1} = r_{3,1,3/2} = r_{3,1,2} = 1$. It follows that $r_{3,n,n+i/2}$ is the coefficient of x^i in the expansion of $(1+x+x^2)^n$, i.e. the trinomial coefficient $N_2(i,n)$.

(vi) When $q = 4$ the recursion of Lemma 3.16(v) is $r_{4,n,s} = r_{4,n-1,s-3} + 2r_{4,n-1,s-2} + r_{3,n-1,s-1}$ and by Lemma 3.16(i) we have $r_{4,1,1} = 1, r_{4,1,2} = 2, r_{4,1,3} = 1$. It follows that $r_{4,n,w}$ is the coefficient of x^{w-n} in the expansion of $(1+2x+x^2)^n = (1+x)^{2n}$, i.e. the binomial coefficient $\binom{2n}{w-n}$. ∎

Note 3.18 *Lemma 3.15 when combined with Corollary 3.17(i) implies that, in the case* $n = 2$, *the* $\mathcal{NOS}_q(2)$ *generated using the approach described in Section 3.3 has optimal period.*

Also, it follows from Corollary 3.17(iii) and (iv) that a $\mathcal{NOS}_q(n)$ *generated using the approach described in Section 3.3 has, by Lemma 3.15, period at least* $(q^n - q^{n-3}(q^2+2))/2 = \frac{q^3-q^2-2}{2q^3}q^n$. *Since* $\frac{q^3-q^2-2}{q^3}$ *approaches* 1 *as* $q \to \infty$, *these negative orientable sequences have period the same order of magnitude as the upper bound. Of course, if* $n > 2$ *the period is unlikely to actually be optimal, as the construction avoids all* n*-tuples of pseudoweight exactly* $nq/2$.

The values of $r_{q,n,nq/2}$ for $q = 3$, $q = 4$ and $1 \leq n \leq 6$ are given in Table 2. These values were derived using Corollary 3.17(v) and (vi).

Table 2: Values of $r_{q,n,nq/2}$ for small q and n

	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$
$q = 3$	3	7	19	51	141	393	1107
$q = 4$	6	20	70	252	924	3432	12870
$q = 5$	5	13	69	261	1301	5685	27525
$q = 6$	8	38	196	1052	5774	32146	180772
$q = 7$	7	19	183	791	6613	35561	267639
$q = 8$	10	62	426	3052	22354	166014	1245066

The periods of the negative orientable sequences using Construction II for $q = 3$, $q = 4$ and $1 \leq n \leq 6$ are given in Table 3. These values were derived using Lemma 3.15 and Table 2.

Table 3: $\mathcal{NOS}_q(n)$ periods (Construction II) for small q and n

	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$
$q = 3$	3	10	31	96	294	897	2727
$q = 4$	5	22	93	386	1586	6476	26333
$q = 5$	10	56	278	1432	7162	36220	181550
$q = 6$	14	89	550	3362	20441	123895	749422
$q = 7$	21	162	1109	8008	55518	393991	2748581
$q = 8$	27	225	1835	14858	119895	965569	7766075

These values can be compared with the bounds given in Table 1.

3.5 Examples of Construction II

We next give some simple examples of the general n construction.

3.5.1 A negative orientable sequence for $n = 2$ and $q = 3$

The nine 3-ary 2-tuples and their pseudoweights are shown in Table 4.

There are clearly just three 2-tuples of pseudoweight less than $nq/2 = 3$, and these can be joined to form the ring sequence

$$S_{3,2} = [110].$$

Table 4: Pseudoweights of q-ary n-tuples: $n = 2$, $q = 3$

u	$w^*(\mathbf{u})$	**u**	$w^*(\mathbf{u})$	**u**	$w^*(\mathbf{u})$
11	2	00	3	02	3.5
01	2.5	12	3	20	3.5
10	2.5	21	3	22	4

This is an optimal period $\mathcal{NOS}_3(2)$ (see Table 1), with $w_3(S_{3,2}) = 2$ (a unit in $\mathbb{Z}_3$)).

3.5.2 A negative orientable sequence for $n = 3$ and $q = 3$

The 27 3-ary 3-tuples and their pseudoweights are shown in Table 5.

Table 5: Pseudoweights of q-ary n-tuples: $n = 3$, $q = 3$

u	$w^*(\mathbf{u})$	**u**	$w^*(\mathbf{u})$	**u**	$w^*(\mathbf{u})$
111	3	211	4	020	5
011	3.5	000	4.5	200	5
101	3.5	012	4.5	122	5
110	3.5	021	4.5	212	5
001	4	102	4.5	221	5
010	4	120	4.5	022	5.5
100	4	201	4.5	202	5.5
112	4	210	4.5	220	5.5
121	4	002	5	222	6

There are ten 3-tuples of pseudoweight less than $nq/2 = 4.5$, and these can be joined to form the ring sequence

$$S_{3,3} = [0100112111].$$

This is a $\mathcal{NOS}_3(3)$ with period one less than the bound of Theorem 3.10 (see Table 1) with $w_3(S_{3,3}) = 2$ (a unit in $\mathbb{Z}_3$).

3.5.3 A negative orientable sequence for $n = 3$ and $q = 4$

The 64 4-ary 3-tuples and their pseudoweights are shown in Table 6.

There are 22 3-tuples of pseudoweight less than $nq/2 = 6$, and these can be joined to form the ring sequence

$$S_{4,3} = [0122\ 1201\ 0210\ 0113\ 1112\ 11]$$

Table 6: Pseudoweights of q-ary n-tuples: $n = 3$, $q = 4$

$\mathbf{u}$	$w^*(\mathbf{u})$	$\mathbf{u}$	$w^*(\mathbf{u})$	$\mathbf{u}$	$w^*(\mathbf{u})$	$\mathbf{u}$	$w^*(\mathbf{u})$
111	3	113	5	022	6	230	7
011	4	131	5	202	6	302	7
101	4	311	5	220	6	320	7
110	4	122	5	123	6	133	7
112	4	212	5	132	6	313	7
121	4	221	5	213	6	331	7
211	4	000	6	231	6	223	7
001	5	002	6	312	6	232	7
010	5	020	6	321	6	322	7
100	5	200	6	222	6	033	8
012	5	013	6	003	7	303	8
021	5	031	6	030	7	330	8
102	5	103	6	300	7	233	8
120	5	130	6	023	7	323	8
201	5	301	6	032	7	332	8
210	5	310	6	203	7	333	9

where the spaces have been added here and in other examples solely to improve readability. This is a $\mathcal{NOS}_4(3)$ with period five less than the bound of Theorem 3.10 (see Table 1) with $w_4(S_{4,3}) = 0$.

3.6 Construction III: A construction avoiding zeros

We now give a further method of construction for negative orientable sequences. This approach yields sequences with a somewhat shorter period than those generated by Construction II. However, as discussed in Section 5, the sequences have a special property that makes them suitable for use with a recursive method of construction for orientable sequences described in [2].

We first need the following definition.

Definition 3.19 *If* $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ *is an* n*-tuple of elements of* $\mathbb{Z}_q$ $(q > 1)$*, then the* weight *of* $\mathbf{u}$ *is defined to be the sum*

$$w(\mathbf{u}) = \sum_{i=0}^{n-1} u_i$$

where we compute the sum in $\mathbb{Z}$ *and treat the values* u_i *as integers in the range* $[0, q-1]$.

We can now state the following elementary result.

Theorem 3.20 *Suppose $n \geq 2$, $q > 2$, and S is a q-ary n-window sequence containing no zeros and with the property that all the n-tuples appearing in S have weight less than $nq/2$. Then S is a $\mathcal{NOS}_q(n)$.*

Proof If $\mathbf{u}$ is an n-tuple in S then $w(-\mathbf{u}^R) = nq - w(\mathbf{u})$ since $\mathbf{u}$ is zero-free. Hence, since $w(\mathbf{u}) < nq/2$, it follows immediately that $w(-\mathbf{u}^R) > nq/2$. Hence the n-tuples in $-S^R$ are all distinct from the n-tuples in S. Moreover, the n-tuples in S (and in $-S^R$) are all distinct because S is an n-window sequence. Hence S is a $\mathcal{NOS}_q(n)$. ∎

It remains to establish how to construct sequences S with the properties of Theorem 3.20. Suppose, for $q > 2$ and $n \geq 2$, G is a directed graph with vertices the zero-free q-ary $(n-1)$-tuples with weight less than $nq/2 - 1$, and with a directed edge connecting vertices $\mathbf{u} = (u_0, u_1, \ldots, u_{n-2})$ and $\mathbf{v} = (v_0, v_1, \ldots, v_{n-2})$ if and only if $u_i = v_{i-1}$, $(1 \leq i \leq n-2)$, and $w(\mathbf{u}) + v_{n-2} < nq/2$.

As previously, we identify the edge connecting $(u_0, u_1, \ldots, u_{n-2})$ to $(u_1, u_2, \ldots, u_{n-2}, v)$ with the n-tuple $(u_0, u_1, \ldots, u_{n-2}, v)$. It follows immediately that every zero-free q-ary n-tuple with weight less than $nq/2$ will appear as an edge in G.

A directed circuit in G will clearly give us a sequence S with the desired property. Consider any vertex $\mathbf{u} = (u_0, u_1, \ldots, u_{n-2})$. An incoming edge

$$(s, u_0, u_1, \ldots, u_{n-2})$$

must satisfy $s + w(\mathbf{u}) < nq/2$. Similarly an outgoing edge

$$(u_0, u_1, \ldots, u_{n-2}, t)$$

must satisfy $t + w(\mathbf{u}) < nq/2$. That is, the in-degree of every vertex is the same as its out-degree. Moreover, G is clearly connected, as it is straightforward to construct a 'low weight' directed path between any pair of vertices (just as in the discussion in Section 3.3). This means there exists a directed Eulerian Circuit in G (see, for example, [12, Corollary 6.1]), yielding a q-ary zero-free n-window sequence S with the property that all the n-tuples appearing in S have weight less than $nq/2$, and with period equal to the number of zero-free q-ary n-tuples with weight less than $nq/2$.

If nq is odd (i.e. if n and q are both odd), then the number of zero-free n-tuples with weight less than $nq/2$ is simply $(q-1)^n/2$ (since for every zero-free n-tuple of weight less than $nq/2$ there is one of weight greater than $nq/2$).

For nq even, the situation is a little more complex. Let $k_{q,n,w}$ ($q \geq 2$, $n \geq 1$) denote the number of zero-free q-ary n-tuples with weight exactly w. Then, if nq is even, the number of n-tuples is $((q-1)^n - k_{q,n,nq/2})/2$.

The above discussion establishes the following result.

Lemma 3.21 *There exists a $\mathcal{NOS}_q(n)$ containing no zeros of period $(q-1)^n/2$ (if q and n are both odd) or $((q-1)^n - k_{q,n,nq/2})/2$ (if q or n is even) for all $q > 2$ and $n \geq 2$.*

We show, in the next subsection, that for $q > 2$, $k_{q,n,nq/2} < (q-1)^{n-1}$. Thus a sequence of $\mathcal{NOS}_q(n)$ exists with period at least $\frac{q-2}{q-1}(q-1)^n/2$. Although this period is not of the same order of magnitude as the upper bound of Theorem 3.10 we note that the sequences constructed in this section are negative orientable sequences of order n over the alphabet $\mathbb{Z}_q\backslash\{0\}$ of size $q-1$ and the period of such sequences is bounded above by $(q-1)^n/2$. Now $\frac{q-2}{q-1} \geq 1/2$ and approaches 1 as $q \to \infty$ so the period is the same order of magnitude as this upper bound. Indeed when nq is odd this upper bound is attained.

As we discuss below, it is of interest to know the weight of the $\mathcal{NOS}_q(n)$ constructed by the above method. We have the following elementary result.

Lemma 3.22 *Suppose S is a $\mathcal{NOS}_q(n)$ constructed according to Theorem 3.20 using a sequence S containing all zero-free q-ary n-tuples with weight less than $nq/2$. Then:*

$$w(S) = \frac{1}{n} \sum_{w=n}^{\lfloor nq/2 \rfloor} w k_{q,n,w}.$$

Proof If we add the weights of all the n-tuples occurring in the ring sequence S then the result will clearly be $nw(S)$. Since S contains all possible zero-free q-ary n-tuples of weight less than $nq/2$, the result follows. ∎

The values of $k_{q,n,nq/2}$ for $q = 3$, $q = 4$ and $1 \leq n \leq 6$ are given in Table 7 (observing that $k_{q,n,nq/2}$ is undefined for q and n both odd). These values were derived using Lemma 3.23.

The periods of the negative orientable sequences using Construction III for $q = 3$, $q = 4$ and $1 \leq n \leq 6$ are given in Table 8. These values were derived using Lemma 3.21 and Table 7.

3.7 Enumerating n-tuples of given weight

Lemmas 3.21 and 3.22 mean that it is of interest to know the value of $k_{q,n,w}$. We have the following elementary result.

Table 7: Values of $k_{q,n,nq/2}$ for $q = 3$ and $q = 4$

	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$q = 3$	2	–	6	–	20
$q = 4$	3	7	19	51	141

Table 8: $\mathcal{NOS}_q(n)$ periods (Construction III) for $q = 3$ and $q = 4$

	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$q = 3$	1	4	5	16	22
$q = 4$	3	10	31	96	294

Lemma 3.23 *As above, suppose $k_{q,n,w}$ ($q \geq 2$, $n \geq 1$) is the number of zero-free q-ary n-tuples with weight exactly w, where $k_{q,n,w} = 0$ by definition if $w < n$ or $w > n(q-1)$.*

(i) $k_{q,1,i} = 1$, *for* $1 \leq i \leq q-1$.

(ii) $\sum_{i=n}^{n(q-1)} k_{q,n,i} = (q-1)^n$;

(iii) *if* $n > 1$, $k_{q,n,w} = \sum_{i=1}^{q-1} k_{q,n-1,w-i}$.

(iv) $k_{q,2,i} = q - 1 - |q - i|$, *for* $2 \leq i \leq 2(q-1)$.

(v) *if q is even,* $k_{q,3,3q/2} = \frac{3q^2-6q+4}{4}$.

Proof Parts (i) and (ii) are trivially true; (iii) follows immediately by considering the $q-1$ possible ways of adding a final entry to a zero-free $(n-1)$-tuple to make a zero-free n-tuple of a given weight. (iv) follows from (i) and (iii). Finally, since

$$\begin{aligned} k_{q,3,3q/2} &= \sum_{i=1}^{q-1} k_{q,2,3q/2-i} \quad \text{(from (iii))} \\ &= \sum_{i=1}^{q-1} q - 1 - |i - q/2| \quad \text{(from (iv))} \end{aligned}$$

and the result follows. ∎

Corollary 3.24 *Suppose $q > 2$.*

(i) *If* $2 \leq w \leq 2q-2$, *then* $k_{q,2,q} = q-1$ *and* $k_{q,2,w} < q-1$ *if* $w \neq q$.

(ii) If $n \geq 3$ and $n \leq w \leq n(q-1)$, then $k_{q,n,w} < (q-1)^{n-1}$.

Proof (i) This follows immediately by Lemma 3.23(iv)).

(ii) By Lemma 3.23(iii),

$$k_{q,3,w} = \sum_{i=1}^{q-1} k_{q,2,i} < (q-1)^2, \text{by (i)}.$$

The result now follows by induction using Lemma 3.23(iii).

The recursion of Lemma 3.23(iii) makes calculating $k_{q,n,w}$ simple for small n and q. In fact these values are well-studied. It follows immediately from Lemma 3.23(i) and (iii) that the values $k_{3,n,w}$, $n \leq w \leq 2n$ are simply the binomial coefficients $\binom{n}{w-n}$. More generally, using the notation of Freund [8], $k_{q,n,w} = N_{q-2}(w-n, n)$, where $N_m(r,k)$ denotes the number of ways in which r identical objects can be distributed in n cells with at most m objects per cell. The values $k_{q,n,w}$ are simply the coefficients of the polynomial $(1 + x + x^2 + \cdots + x^{q-2})^n$. The values of $N_m(r,k)$ are sometimes referred to as trinomial and quadrinomial coefficients in the cases $m = 2$ and 3, and Comtet [6, page 78] tabulates the values of the trinomial and quadrinomial coefficients for small r and k.

3.8 Examples of Construction III

3.8.1 Examples for $q = 3$

For $q = 3$ and small n, the zero-free low-weight tuples used in the construction of Section 3.6 are shown in Tables 9 and 10.

Table 9: Zero-free q-ary n-tuples of weight less than $qn/2$: $n = 2$, $q = 3$

$\mathbf{u}$	$w(\mathbf{u})$
11	2

Table 10: Zero-free q-ary n-tuples of weight less than $qn/2$: $n = 3$, $q = 3$

$\mathbf{u}$	$w(\mathbf{u})$	$\mathbf{u}$	$w(\mathbf{u})$	$\mathbf{u}$	$w(\mathbf{u})$	$\mathbf{u}$	$w(\mathbf{u})$
111	3	112	4	121	4	211	4

The single 2-tuple in Table 9 and the four 3-tuples in Table 10 give rise to a $\mathcal{NOS}_3(2)$ of period 1 with ring sequence [1], and a $\mathcal{NOS}_3(3)$ of period 4 with ring sequence [1112], respectively.

3.8.2 Examples for $q = 4$

For $q = 4$ and small n, the zero-free low-weight tuples are shown in Tables 11 and 12.

Table 11: Zero-free q-ary n-tuples of weight less than $qn/2$: $n = 2$, $q = 4$

$\mathbf{u}$	$w(\mathbf{u})$	$\mathbf{u}$	$w(\mathbf{u})$	$\mathbf{u}$	$w(\mathbf{u})$
11	2	12	3	21	3

Table 12: Zero-free q-ary n-tuples of weight less than $qn/2$: $n = 3$, $q = 4$

$\mathbf{u}$	$w(\mathbf{u})$	$\mathbf{u}$	$w(\mathbf{u})$	$\mathbf{u}$	$w(\mathbf{u})$	$\mathbf{u}$	$w(\mathbf{u})$	$\mathbf{u}$	$w(\mathbf{u})$
111	3	112	4	121	4	211	4	113	5
131	5	311	5	122	5	212	5	221	5

The three 2-tuples in Table 11 and the ten 3-tuples in Table 12 give rise to a $\mathcal{NOS}_4(2)$ of period 3 with ring sequence [112], and a $\mathcal{NOS}_4(3)$ of period 10 with ring sequence [1113112212], respectively.

4 Orientable sequences

We now describe a way of using the negative orientable sequences generated by Construction II to yield orientable sequences for general n and q with period of the same order of magnitude as the upper bound. We first need to introduce the Lempel Homorphism, which is fundamental to the construction of orientable sequences.

4.1 The de Bruijn graph and the Lempel Homomorphism

Following Alhakim et al. [2] we also introduce the de Bruijn graph. For positive integers n and q greater than one, let $\mathbb{Z}_q^n$ be the set of all q^n vectors of length n with entries from the group $\mathbb{Z}_q$ of residues modulo q. A de Bruijn sequence *of order* n with alphabet in $\mathbb{Z}_q$ is a periodic sequence that includes every possible n-tuple precisely once as a subsequence of consecutive symbols in one period of the sequence.

The order n de Bruijn digraph, $B_n(q)$, is a directed graph with $\mathbb{Z}_q^n$ as its vertex set and where, for any two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, $(\mathbf{x}; \mathbf{y})$ is an edge if and only if $y_i = x_{i+1}$ for every i $(1 \leq i < n)$. We then say that $\mathbf{x}$ is a *predecessor* of $\mathbf{y}$ and $\mathbf{y}$ is a *successor* of $\mathbf{x}$. Evidently, every vertex has exactly q successors and q predecessors.

A cycle in $B_n(q)$ is a path that starts and ends at the same vertex. It is said to be *vertex disjoint* if it does not visit any vertex more than once. Two cycles or two paths in the digraph are vertex-disjoint if they do not have a common vertex.

Following the notation of Lempel [13], a convenient representation of a vertex disjoint cycle $(\mathbf{x}^{(1)}; \ldots; \mathbf{x}^{(l)})$ is the *ring sequence* $[x^1, \ldots, x^l]$ of symbols from $\mathbb{Z}_q$ defined such that the ith vertex in the cycle starts with the symbol x^i. Clearly a vertex disjoint cycle with ring sequence $[x_1, x_2, \ldots, x_m]$ corresponds to an n-window sequence of period m where $s_i = x_j$ whenever $i \equiv j \pmod{m}$.

Finally, we need a well-established generalisation of the Lempel graph homomorphism [13] to non-binary alphabets — see, for example, Alhakim and Akinwande [1].

Definition 4.1 *For a nonzero $\beta \in \mathbb{Z}_q$, we define a function D_β from $B_n(q)$ to $B_{n-1}(q)$ as follows. For $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_{n-1})$, $D_\beta(\boldsymbol{a}) = \boldsymbol{b}$ if and only if $b_i = d_\beta(a_i, a_{i+1})$ for $i = 1$ to $n-1$, where $d_\beta(a_i, a_{i+1}) = \beta(a_{i+1} - a_i) \mod q$.*

We extend the notation to allow the Lempel morphism D_β to be applied to periodic sequences in the natural way, as we now describe. That is, D_β (where $\beta \in \mathbb{Z}_q$) is the map from the set of periodic sequences to itself defined by

$$D((s_i)) = \{(t_i) : t_j = \beta(s_{j+1} - s_j)\}.$$

The image of a sequence of period m will clearly have period dividing m. In the usual way we can define D_β^{-1} to be the *inverse* of D_β, i.e. if S is a periodic sequence than $D_\beta^{-1}(S)$ is the set of all sequences T with the property that $D_\beta(T) = S$.

We are particularly interested in the case $\beta = 1$, and we simply write D for D_1. Examples of the application of D^{-1} are given in Sections 4.5 and 5.3.

The *weight* $w(S)$ of a cycle S is the weight of the ring sequence corresponding to S (that is the sum of the terms $s_0, s_1, \ldots, s_{m-1}$ treating s_i as an integer in the range $[0, q-1]$). Similarly we write $w_q(S)$ for $w(S) \bmod q$.

4.2 Construction methods

We need the following key result from Alhakim et al. [2].

Result 4.2 (Theorem 6.10 of [2]) *Suppose $S = (s_i)$ is a negative orientable sequence of order n and period m. If $w_q(S)$ has additive order h as a residue modulo q then $D^{-1}(S)$ consists of h shifts of each of q/h mutually no-disjoint orientable sequences of order $n+1$ and period hm which are translates of one another.*

So, in particular, if a $\mathcal{NOS}_q(n)$ S has period m and $(w_q(S), q) = 1$ (i.e. $w_q(S)$ is a unit in $\mathbb{Z}_q$) then $D^{-1}(S)$ is an $\mathcal{OS}_q(n+1)$ of period qm. We propose to employ the negative orientable sequences described in Sections 3.2 and 3.3 to generate orientable sequences using this approach. Unfortunately, we do not know in general which of the negative orientable sequences have weight a unit in $\mathbb{Z}_q$; however, regardless of their actual weight, we next describe a very simple way of modifying these sequences to achieve the desired result.

We first make a simple observation regarding units in $\mathbb{Z}_q$ for any q (where a unit $u \in \mathbb{Z}_q$ is any value such that $(u, q) = 1$).

Lemma 4.3 *Suppose $q > 2$ ($q \neq 6$) and $w \in \mathbb{Z}_q$ is a non-unit, i.e. $(w, q) \neq 1$. Then there exists $d \in \{1, 2, \ldots, \lfloor (q-1)/2 \rfloor\}$ such that $w - d$ is a unit.*

Proof If q is prime then this follows immediately since every non-zero element in $\mathbb{Z}_q$ is a unit. If $q = 4$, then w must be 0 or 2 and in both cases $w - 1$ is a unit. Hence suppose $q > 6$.

We first show that there exists a unit u satisfying $q/2 < u \leq 3q/4$. If q is odd, $q = 2s + 1$ say ($s \geq 3$), then trivially $s + 1$ is a unit in $\mathbb{Z}_{2s+1}$ since $(s+1, 2s+1) = 1$ for every $s > 0$, and $s+1$ is within the desired range. If q is even, first suppose $q \equiv 0 \pmod 4$, i.e. $q = 4s$ for some $s \geq 2$. Then $2s+1$ is a unit in $\mathbb{Z}_{4s}$ since $(2s + 1, 4s) = 1$ for every $s > 0$, and $2s + 1$ is within the desired range. Finally suppose $q \equiv 2 \pmod 4$, i.e. $q = 4s + 2$ for some $s \geq 2$. Then $2s + 3$ is a unit in $\mathbb{Z}_{4s+2}$ since $(2s + 3, 4s + 2) = 1$ for every $s > 0$, and $2s + 3$ is within the desired range since $s \geq 2$.

If $w = 0$ then $w - 1$ is a unit and if $0 < w \leq q/2$ then clearly there exists $d \in \{1, 2, \ldots, \lfloor (q-1)/2 \rfloor\}$ such that $w - d = 1$, and the result follows. Hence suppose $q/2 < w < q - 1$. Since $q > 6$ there exists a unit u satisfying $q/2 < u \leq 3q/4$. If $u < w$ then clearly there will exist a $d \in \{1, 2, \ldots, \lfloor (q-1)/2 \rfloor\}$ such that $w - d = u$. If $w < u$ then there exists a $d \in \{1, 2, \ldots, \lfloor (q-1)/2 \rfloor\}$ such that $w - d = -u$; this follows since $w < u \leq 3q/4$ and $-u \geq q/4$. The result follows by noting that $-u$ must be a unit. ∎

As is well-known, it is possible to remove a uniform n-tuple from a sequence without changing the other n-tuples present, simply by deleting a single occurrence of a symbol from within the uniform n-tuple. That is, if S is a $\mathcal{NOS}_q(n)$ of period m containing the i-uniform n-tuple $(i, i, \ldots, i)$, deleting a single occurrence of i from within this tuple will result in a $\mathcal{NOS}_q(n)$ S' of period $m-1$, where $w_q(S') = w_q(S) - i$. This leads to the following simple construction.

Construction 4.4 *Suppose $q > 2$ and let S be a $\mathcal{NOS}_q(n)$ with the property that it contains all the i-uniform n-tuples $(i, i, \ldots, i)$ for every i satisfying $1 \leq i < q/2$. We derive a sequence S' from S in the following way.*

- *If $w_q(S)$ is a unit in $\mathbb{Z}_q$ then set $S' = S$.*
- *If $w_q(S)$ is not a unit in $\mathbb{Z}_q$ and $q \neq 6$, choose $i \in [1, q/2)$ such that $w_q(S) - i$ is a unit (i exists by Lemma 4.3). By assumption, S contains the i-uniform n-tuple $(i, i, \ldots, i)$, and let S' be derived from S by deleting a single occurrence of i from this n-tuple.*
- *If $w_q(S)$ is not a unit in $\mathbb{Z}_q$ and $q = 6$, then $w_q(S)$ must be one of 0, 2, 3 and 4. Observe that S must contain the uniform n-tuples $(1, 1, \ldots, 1)$ and $(2, 2, \ldots, 2)$. If $w_q(S) = 0$ or 2, let S' be derived from S by deleting a single occurrence of 1 from the 1-uniform n-tuple $(1, 1, \ldots, 1)$. If $w_q(S) = 3$, let S' be derived from S by deleting a single occurrence of 2 from $(2, 2, \ldots, 2)$. If $w_q(S) = 4$ let S' be derived from S by deleting a single occurrence of 1 from $(1, 1, \ldots, 1)$ and a single occurrence of 2 from $(2, 2, \ldots, 2)$.*

Lemma 4.5 *Suppose that $q > 2$, S is a $\mathcal{NOS}_q(n)$ of period m containing all the i-uniform n-tuples $(i, i, \ldots, i)$ for every i satisfying $1 \leq i < q/2$, and S' is derived from S using Construction 4.4. Then S' is a $\mathcal{NOS}_q(n)$ with $w_q(S')$ a unit in $\mathbb{Z}_q$, and period at least $m-1$ ($q \neq 6$) or at least $m-2$ ($q = 6$).*

Proof Observe that the construction of S' in the case $q = 6$ ensures that $w_6(S')$ is a unit. The result follows trivially by the observation that removing a single element from a uniform n-tuple will not affect negative orientability. ∎

4.3 Orientable sequences for $n = 3$

We can now give a method for constructing near-optimal orientable sequences for $n = 3$ and any $q > 2$. These have period asymptotical of the same order as the maximum possible as $q \to \infty$.

Construction 4.6 *Suppose $q > 2$ and let S be a maximal $\mathcal{NOS}_q(2)$ constructed using the method described in Section 3.2. Let S' be derived from S using Construction 4.4 (observing that S satisfies the conditions of the construction by Note 3.12). Then let T be derived from S' using Theorem 4.2.*

Theorem 4.7 *Suppose $q > 2$ and suppose that T is constructed from S using Construction 4.6. Then T is an $\mathcal{OS}_q(3)$ of period at least:*

$$q\left(\frac{q(q-1)}{2} - 1\right) \quad \text{if } q \text{ is odd;}$$
$$q\left(\frac{q(q-1)}{2} - 2\right) \quad \text{if } q \text{ is even } (q \neq 6);$$
$$q\left(\frac{q(q-1)}{2} - 3\right) \quad \text{if } q = 6.$$

Proof By the discussions in Section 3.2, S has period either $q(q-1)/2$ if q is odd or $q(q-1)/2 - 1$ if q is even. Hence, by Lemma 4.5 S' has period at least:

- $q(q-1)/2 - 1$ if q is odd;
- $q(q-1)/2 - 2$ if q is even;
- $q(q-1)/2 - 3$ if $q = 6$.

Applying Theorem 4.2 multiplies the period by q, since in every case $w_q(S')$ is a unit, and the result follows. ∎

Periods of the orientable sequences obtained using Theorem 4.7 are tabulated for small q in Table 13. The value of the upper bound given in [2] is given in brackets for comparison purposes.

Table 13: $\mathcal{OS}_q(3)$ periods (and bounds)

$q=3$	$q=4$	$q=5$	$q=6$	$q=7$	$q=8$
6 (9)	16 (22)	45 (50)	72 (87)	140 (147)	208 (220)

4.4 Orientable sequences for general n

We can apply an almost identical approach to the negative orientable sequences described in Section 3.3. First observe that if a $\mathcal{NOS}_q(n)$ is constructed as given in Section 3.3, then it will contain the i-uniform n-tuples $(i, i, \ldots, i)$ for every i satisfying $1 \leq i < q/2$.

Construction 4.8 *Suppose $q > 2$ and $n > 2$, and let S be a $\mathcal{NOS}_q(n-1)$ constructed using the method described in Section 3.3. Let S' be derived from S using Construction 4.4 (observing that S satisfies the conditions of the construction as noted above). Then let T be derived from S' using Result 4.2.*

Theorem 4.9 *Suppose $n > 2$, $q > 2$, and T is derived using Construction 4.8. Then T is an $\mathcal{OS}_q(n)$ of period at least:*

$$q(q^{n-1} - r_{q,n-1,(n-1)q/2} - 2)/2 \quad \text{if } q \neq 6;$$
$$q(q^{n-1} - r_{q,n-1,(n-1)q/2} - 4)/2 \quad \text{if } q = 6;$$

where, as previously, $r_{n-1,q,s}$ is the number of q-ary $(n-1)$-tuples with pseudoweight exactly s.

Proof By Lemma 3.15 there exists a $\mathcal{NOS}_q(n-1)$ S of period $(q^{n-1} - r_{n,q-1})/2$. By Lemma 4.5, constructing S' from S using Construction 4.4 will yield a $\mathcal{NOS}_q(n-1)$ of period:

- $\frac{q^{n-1}-r_{q,n-1,(n-1)q/2}}{2} - 1$ if $q \neq 6$; and
- $\frac{q^{n-1}-r_{q,n-1,(n-1)q/2}}{2} - 2$ if $q = 6$.

The result follows from Result 4.2. ∎

By Note 3.18, these sequences have period of the same order of magnitude as the upper bound. Periods of the orientable sequences obtained using Theorem 4.9 are tabulated for small q in Table 14. As previously, the value of the upper bound given in [2] is given in brackets for comparison purposes.

4.5 Examples

We next show how the three examples of negative orientable sequences given in Section 3.5 can be used in the above construction method to yield orientable sequences.

As noted above, the $\mathcal{NOS}_3(2)$ $S_{3,2} = [011]$ has $w_3(S_{3,2}) = 2$, a unit in $\mathbb{Z}_3$ and hence can be used directly in Result 4.2. Applying D^{-1} to $S_{3,2}$ results in the $\mathcal{OS}_3(3)$ with ring sequence [001220112], of period 9. This is optimal (see [2, Table 1]).

The $\mathcal{NOS}_3(3)$ $S_{3,3} = [0100112111]$ has $w_3(S_{3,3}) = 2$, which is again a unit in $\mathbb{Z}_3$ and hence it too can be used directly in Result 4.2. Applying D^{-1} to $S_{3,3}$ results in the $\mathcal{OS}_3(4)$ with ring sequence

[0011120201 2200012120 1122201012],

Table 14: $\mathcal{OS}_q(n)$ periods (and bounds)

n	$q=3$	$q=4$	$q=5$	$q=6$	$q=7$	$q=8$
3	6	16	45	72	140	208
	(9)	(22)	(50)	(87)	(147)	(220)
4	27	84	275	522	1127	1792
	(33)	(118)	(290)	(627)	(1155)	(2012)
5	90	368	1385	3288	7756	14672
	(105)	(478)	(1490)	(3777)	(8211)	(16124)
6	285	1540	7155	20160	56049	118856
	(336)	(2014)	(7680)	(23217)	(58464)	(130812)
7	879	6340	35805	122634	388619	959152
	(1032)	(8062)	(38640)	(139317)	(410256)	(1046524)
8	2688	25900	181095	743358	2757930	7724544
	(3189)	(32638)	(194630)	(839157)	(2879835)	(8386556)

of period 30. This is three less than the bound of [2, Theorem 4.11].

Finally, the $\mathcal{NOS}_4(3)$ $S_{4,3}$ has $w_4(S_{4,3}) = 0$, which is not a unit in $\mathbb{Z}_4$. To use it in Result 4.2 we need to remove a single '1' from the 1-uniform 3-tuple (111), resulting in the $\mathcal{NOS}_4(3)$

$$S'_{4,3} = [0122\ 1201\ 0210\ 0113\ 1121\ 1]$$

with period 21 and $w_4(S'_{4,3}) = 3$, a unit in $\mathbb{Z}_4$. Applying D^{-1} to $S'_{4,3}$ results in the $\mathcal{OS}_4(4)$ with ring sequence:

$$[0013\ 1200\ 1130\ 0012\ 1231\ 2\ \ 3302\ 0133\ 0023\ 3301\ 0120\ 1$$
$$2231\ 3022\ 3312\ 2230\ 3013\ 0\ \ 1120\ 2311\ 2201\ 1123\ 2302\ 3],$$

of period 84. This is 34 less then the bound of [2, Theorem 4.11].

5 More orientable sequences

In this final main section we describe how to construct orientable sequences using a second approach described by Alhakim et al. [2]. These have period of the same order of magnitude as the upper bound.

5.1 Preliminaries

We first need the following definitions.

Definition 5.1 (Definition 6.16 of [2]) *Let $S = (s_i)$ be a periodic q-ary sequence. For $a \in \mathbb{Z}_q$ we write a^t for a string of t consecutive terms a of S. A* run *of a in S is a string $s_j, \ldots, s_{j+t-1} = a^t$ with $s_{j-1}, s_{j+t} \neq a$. A run, a^t, is* maximal *in S if any string $a^{t'}$ of S has $t' \leq t$ and further is* inverse maximal *if, in addition, any string $1^{t'} + (q-1-a)^{t'}$ of S has $t' \leq t$.*

We note that if a^t is a maximal run of S then $(a+b)^t$ is a maximal run of the translate by b of S and there exists a shift of such a translate that has ring sequence whose first t terms are $a+b$. Moreover if S is orientable (respectively negative orientable) then so is this shifted translate.

Definition 5.2 (Definition 6.17 of [2]) *Let $a \in \mathbb{Z}_q$. Suppose that the ring sequence of a periodic sequence S is $[s_0, s_1, \cdots, s_{m-1}]$ and r is the smallest non-negative integer such that $a^t = s_r, s_{r+1}, \cdots, s_{r+t-1}$ is a maximal run for $a \in \mathbb{Z}_q$. Define the sequence $\mathcal{E}_a(S)$ to be the sequence with ring sequence*

$$[s_0, s_1, \cdots, s_{r-1}, a, s_r, s_{r+1}, \cdots, s_{m-1}]$$

i.e. where the occurrence of a^t is replaced with a^{t+1}.

We also need:

Definition 5.3 (Definition 6.20 of [2]) *An orientable (respectively negative orientable) sequence with the property that any run of 0 has length at most $n-2$ is said to be* good.

We can now give the following result.

Result 5.4 *[Corollary 6.22 of [2]] Suppose S_n is a good orientable (respectively negative orientable) sequence of order n and period m_n with the property that $w_q(S)$ is a unit in $\mathbb{Z}_q$ ($n \geq 2$ and $q > 2$). Recursively define the sequences $S_{i+1} = \mathcal{E}_a(D^{-1}(S_i))$, where $a = 1 - w_q(D^{-1}(S_i))$, for $i \geq n$, and suppose S_i has period m_i ($i > n$). Then, S_i is a good orientable or negative orientable sequence for every $i \geq n$, and $m_{n+j+1} = qm_{n+j} + 1$ for every $j \geq 0$. S_i is an orientable (respectively negative orientable) sequence when $i - n$ is even and a negative orientable (respectively orientable) sequence when $i - n$ is odd.*

Note 5.5 *Corollary 6.22 of [2] differs slightly from the above statement in that it makes the slightly more restrictive assumption that $w_q(S) = 1$. However, it is straightforward to verify that the result holds whenever $w_q(S)$ is a unit in $\mathbb{Z}_q$.*

The following simple result follows.

Corollary 5.6 *In the notation of Result 5.4*

$$m_{n+s} = q^s m_n + \frac{(q^s - 1)}{(q - 1)}$$

for every $s \geq 1$.

Proof We establish the result by induction on s. Suppose $s = 1$; then from Result 5.4 we immediately know that

$$m_{n+1} = q m_n + 1$$

i.e. the desired result holds in this case since, for $s = 1$:

$$q^s m_n + \frac{(q^s - 1)}{(q - 1)} = q m_n + \frac{(q - 1)}{(q - 1)} = q m_n + 1.$$

Now suppose the results holds for $s = t > 0$. That is, suppose

$$m_{n+t} = q^t m_n + \frac{(q^t - 1)}{(q - 1)}.$$

By Result 5.4,

$$m_{n+t+1} = q(q^t m_n + \frac{(q^t - 1)}{(q - 1)}) + 1 = q^{t+1} m_n + \frac{(q^{t+1} - q)}{(q - 1)}) + 1$$

and the result clearly holds for $t + 1$. ∎

This immediately gives.

Corollary 5.7 *Suppose* S_n *is a good* $\mathcal{NOS}_q(n)$ *of period* m *with the property that* $w_q(S)$ *is a unit in* $\mathbb{Z}_q$*. Then* S_{n+2s+1} *is a good* $\mathcal{OS}_q(n+2s+1)$ *of period* ℓ_s *for every* $s > 0$*, where*

$$\ell_s = q^{2s+1} m + \frac{(q^{2s+1} - 1)}{(q - 1)}.$$

5.2 Recursively constructing orientable sequences

Combining Result 5.4 and Corollary 5.7 with Construction III, as given in Section 3.6, yields the following result.

Theorem 5.8 *If* $q > 2$*, there exists a:*

- *good* $\mathcal{OS}_q(2s+3)$ *of period at least*

$$q^{2s+1}m_2 + \frac{(q^{2s+1}-1)}{(q-1)}$$

- *good* $\mathcal{OS}_q(2s+4)$ *of period at least*

$$q^{2s+1}m_3 + \frac{(q^{2s+1}-1)}{(q-1)}$$

for every $s \geq 0$*, where*

$$m_2 = \begin{cases} (q-1)(q-2)/2-1, & \text{if } q \neq 6 \\ (q-1)(q-2)/2-2, & \text{if } q = 6 \end{cases}$$

and

$$m_3 = \begin{cases} (q-1)^3/2-1, & \text{if } q \text{ is odd} \\ ((q-1)^3-(3q^2-6q+4)/4)/2-1, & \text{if } q \text{ is even } (q \neq 6) \\ ((q-1)^3-(3q^2-6q+4)/4)/2-2, & \text{if } q = 6. \end{cases}$$

Proof First observe that if S is a negative orientable sequence obtained using the method underlying Lemma 3.21 then it is good by definition (since it contains no zeros). It also clearly contains all the i-uniform n-tuples $(i, i, \ldots, i)$ for every i satisfying $1 \leq i < q/2$. If it has period m then, using Construction 4.4, we can construct a good $\mathcal{NOS}_q(n)$ S' where $w(S')$ is a unit in $\mathbb{Z}_q$, with period at least $m-1$ (or $m-2$ if $q=6$).

- From Lemma 3.21 with $n=2$ there exists a good $\mathcal{NOS}_q(2)$ of period $((q-1)^2-(q-1))/2 = (q-1)(q-2)/2$ (observing that $k_{q,2,q} = q-1$, from Lemma 3.23(iv)). Hence from the above discussion there exists a good $\mathcal{NOS}_q(2)$ whose weight is a unit in $\mathbb{Z}_q$ with period at least $(q-1)(q-2)/2-1$ if $q \neq 6$ and period at least $(q-1)(q-2)/2-2$ if $q=6$.

- From Lemma 3.21 with $n=3$ there exists a good $\mathcal{NOS}_q(3)$ of period $(q-1)^3/2$ if q is odd and period $((q-1)^3-k_{q,3,3q/2})/2$ if q is odd. Hence from the above discussion there exists a good $\mathcal{NOS}_q(3)$ whose weight is a unit in $\mathbb{Z}_q$ with period at least $(q-1)^3/2-1$ if q is odd, $((q-1)^3-k_{q,3,3q/2})/2-1$ if q is even $(q \neq 6)$ and period at least $((q-1)^3-k_{q,3,3q/2})/2-2$ if $q=6$. The result follows by noting that $k_{q,3,3q/2})/2 = (3q^2-6q+4)/4$, from Lemma 3.23(v). ∎

Thus if, for example, q is odd, there exists a good $\mathcal{OS}_q(2s+4)$ of period at least

$$q^{2s+1}(q-1)^3/2 - 1 + \frac{(q^{2s+1}-1)}{(q-1)}.$$

For each s these have period of the same order of magnitude as the upper bound.

5.3 Examples

We conclude by giving two simple examples of the approach described.

- **Generating a good $\mathcal{OS}_3(2i)$ for every $i \geq 2$**

  From Section 3.8.1 we know that $S_3 = [1112]$ is a good $\mathcal{NOS}_3(3)$ of period 4 where $w_3(S_3) = 2$, a unit in $\mathbb{Z}_3$. Applying Result 5.4,

  $$D^{-1}(S_3) = [012020121201]$$

  and

  $$S_4 = \mathcal{E}_1(D^{-1}(S_3)) = [0120201212011]$$

  is a good $\mathcal{OS}_3(4)$ of period 13 and $\mathbb{Z}_3$-weight 1. This recursive process can now be repeated to obtain a good $\mathcal{OS}_3(2i)$ with unit $\mathbb{Z}_3$-weight for every $i > 1$.

- **Generating a good $\mathcal{OS}_4(2i+1)$ for every $i \geq 1$**

  From Section 3.8.1 we know that [112] is a good $\mathcal{NOS}_4(2)$ of period 3 with $\mathbb{Z}_4$-weight 0. We can delete a single 1 to obtain $T_2 = [12]$, a good $\mathcal{NOS}_4(2)$ of period 2 with $\mathbb{Z}_4$-weight 3, a unit in $\mathbb{Z}_4$. Applying Result 5.4,

  $$D^{-1}(T_2) = [01302312]$$

  and

  $$T_3 = \mathcal{E}_1(D^{-1}(T_2)) = [011302312]$$

  is a good $\mathcal{OS}_4(3)$ of period 9 and $\mathbb{Z}_4$-weight 1. This recursive process can now be repeated to obtain a good $\mathcal{OS}_4(2i+1)$ with unit $\mathbb{Z}_4$-weight for every $i > 0$.

6 Concluding remarks

In this paper we used two of the approaches proposed in [2] to generate orientable sequences with period of the same order of magnitude as the upper bound. Whilst the second approach yields sequences with somewhat shorter periods, in practical applications it may be that its recursive (and explicit) approach is more convenient.

There remain a variety of directions for future research. The constructions in this paper require an exponential amount of space with respect to n. They either require storage of a graph of exponential size to use an Eulerian circuit algorithm, or storage of the sequence of a smaller order to use Lempel's homomorphism. It would be desirable to be able to generate negative orientable sequences more space-efficiently. Another potential future research direction would be to generate longer negative orientable sequences by incorporating some words with pseudoweight equal to $nq/2$. Further, it remains to consider how to devise potential input sequences for the other methods of construction proposed in [2].

A related research problem relates to position location applications. Orientability guarantees that viewing an n-tuple uniquely locates the position and direction of reading within a period of a sequence. As a result it would be desirable to have an efficient algorithm for mapping from an n-tuple to its location and orientation, where efficient here means more space-efficient than a large look-up table and more time-efficient than searching through the sequence. Such problems, sometimes referred to as decoding problems, have been previously studied for related classes of sequences and arrays; indeed, Chung, Diaconis and Graham [5] listed this as a fundamental question for the study of de Bruijn sequences.

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