Special orientable sequences

Chris J. Mitchell and Peter R. Wild
Information Security Group, Royal Holloway, University of London, UK
me@chrismitchell.net; peterrwild@gmail.com

17th January 2026

Abstract

Analogously to de Bruijn sequences, orientable sequences have application in automatic position-location applications and, until recently, studies of these sequences focused on the binary case. In recent work by Alhakim et al., recursive methods of construction were described for orientable sequences over arbitrary finite alphabets, requiring ‘starter sequences’ with special properties. Some of these methods required as input special orientable sequences, i.e. orientable sequences which were simultaneously negative orientable. We exhibit methods for constructing special orientable sequences with properties appropriate for use in two of the recursive methods of Alhakim et al. As a result we are able to show how to construct special orientable sequences for arbitrary sizes of alphabet (larger than a small lower bound) and for all window sizes. These sequences have periods asymptotic to the optimal as the alphabet size increases.

1 Introduction

Orientable sequences, i.e. periodic sequences with elements drawn from a finite alphabet with the property that any subsequence of n consecutive elements (an n-tuple) occurs at most once *in either direction*, were introduced in 1992 [3, 4]. They are of interest due to their application in certain position-resolution scenarios. For the binary case, a construction and an upper bound on the period were established by Dai et al. [4], and further constructions were established by Gabrić and Sawada [6] and Mitchell and Wild [8]. A bound on the period and methods of construction for q-ary alphabet sequences (for arbitrary q) were given by Alhakim et al. [2].

In this paper we examine a particular class of orientable sequences known as *special orientable sequences*; such sequences were defined by Alhakim et al. [2], who described a series of recursive constructions for orientable sequences using special orientable sequences as input. We give a bound on the length of special orientable sequences and describe various methods of construction. We then show how certain of the constructed sequences can be used to obtain orientable sequences using methods defined in [2].

1.1 Basic terminology

We first establish some simple notation, following [2]. For mathematical convenience we consider the elements of a sequence to be elements of $\mathbb{Z}_q$ for an arbitrary integer $q > 1$.

For a sequence $S = (s_i)$ we write $\mathbf{s}_n(i) = (s_i, s_{i+1}, \ldots, s_{i+n-1})$. Since we are interested in tuples occurring either forwards or backwards in a sequence we also introduce the notion of a reversed tuple, so that if $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is a q-ary n-tuple (a string of symbols of length n) then $\mathbf{u}^R = (u_{n-1}, u_{n-2}, \ldots, u_0)$ is its *reverse*. The *negative* of a q-ary n-tuple $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ is the n-tuple $-\mathbf{u} = (-u_0, -u_1, \ldots, -u_{n-1})$.

We can then give the following.

Definition 1.1 ([2])**.** A q-ary *n-window sequence* $S = (s_i)$ is a periodic sequence of elements from $\mathbb{Z}_q$ ($q > 1$, $n > 1$) with the property that no n-tuple appears more than once in a period of the sequence, i.e. with the property that if $\mathbf{s}_n(i) = \mathbf{s}_n(j)$ for some i, j, then $i \equiv j \pmod{m}$ where m is the period of the sequence.

Definition 1.2 ([2])**.** An n-window sequence $S = (s_i)$ is said to be an orientable sequence of order n (an $\mathcal{OS}_q(n)$) if $\mathbf{s}_n(i) \neq \mathbf{s}_n(j)^R$, for any i, j.

We also need two related definitions.

Definition 1.3 ([2])**.** An n-window sequence $S = (s_i)$ is said to be a *negative orientable sequence of order* n (a $\mathcal{NOS}_q(n)$) if $\mathbf{s}_n(i) \neq -\mathbf{s}_n(j)^R$, for any i, j.

Definition 1.4 ([2])**.** An orientable sequence $S = (s_i)$ of order n is said to be a *special orientable sequence of order* n (a $\mathcal{SOS}_q(n)$) if, for any i, j, $\mathbf{s}_n(i) \neq -\mathbf{s}_n(j)^R$, i.e. it is also negative orientable.

As discussed in Alhalkim at al. [2], it turns out that negative and special orientable sequences are of importance in constructing orientable sequences. Observe that a sequence is orientable if and only if it is negative orientable for the case $q = 2$. Also note that if $S = (s_i)$ is orientable, negative orientable or special orientable then so is its negative $(-s_i)$.

Bounds on the period of, and methods of construction for, negative orientable sequences were given by Mitchell and Wild [9]; they also showed how to use the constructed negative orientable sequences to construct families of orientable sequences employing two approaches defined in [2]. By contrast, in this paper we focus on special orientable sequences, giving a period bound and methods of construction.

1.2 The de Bruijn graph and the Lempel Homomorphism

Following Alhakim et al. [2] we also introduce the de Bruijn graph. For positive integers n and q greater than one, let $\mathbb{Z}_q^n$ be the set of all q^n vectors of length n with entries from the group $\mathbb{Z}_q$ of residues modulo q. A de Bruijn sequence *of order* n with alphabet in $\mathbb{Z}_q$ is a periodic sequence that includes every possible n-tuple precisely once as a subsequence of consecutive symbols in one period of the sequence.

The order n de Bruijn digraph, $B_n(q)$, is a directed graph with $\mathbb{Z}_q^n$ as its vertex set and where, for any two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, $(\mathbf{x}; \mathbf{y})$ is an edge if and only if $y_i = x_{i+1}$ for every i $(1 \leq i < n)$. We then say that $\mathbf{x}$ is a *predecessor* of $\mathbf{y}$ and $\mathbf{y}$ is a *successor* of $\mathbf{x}$. Evidently, every vertex has exactly q successors and q predecessors. Furthermore, two vertices are said to be *conjugates* if they have the same set of successors.

A cycle in $B_n(q)$ is a path that starts and ends at the same vertex. It is said to be *vertex disjoint* if it does not visit any vertex more than once. Two cycles or two paths in the digraph are vertex-disjoint if they do not have a common vertex. This terminology departs somewhat from standard graph theoretic terminology where the term *closed path* is typically used for what we call a cycle, and *cycle* is used where we use vertex-disjoint cycle.

Following the notation of Lempel [7], a convenient representation of a vertex disjoint cycle $(\mathbf{x}^{(1)}; \ldots; \mathbf{x}^{(l)})$ is the *ring sequence* $[x^1, \ldots, x^l]$ of symbols from $\mathbb{Z}_q$ defined such that the ith vertex in the cycle starts with the symbol x^i. Corresponding to the ring sequence $[x^1, \ldots, x^l]$ is an n-window sequence $S = (s_i)$ where $s_{i+tl} = x_{i+1}$ for $i = 0, \ldots, l-1$ and $t \geq 0$. Conversely, an n-window sequence determines a ring sequence of a vertex disjoint cycle. A *translate* of a word $\mathbf{x} = (x_1, \ldots, x_n)$ is a word $\mathbf{x} + \lambda = (x_1 + \lambda, \ldots, x_n + \lambda)$ where λ is any nonzero element in $\mathbb{Z}_q$ and addition is performed in $\mathbb{Z}_q$. We also define a translate of a cycle as the cycle obtained by a translate of the ring sequence that defines this cycle.

Finally, we need a well-established generalisation of the Lempel graph homomorphism [7] to non-binary alphabets — see, for example, Alhakim and Akinwande [1] (in fact we use a simplified version of their definition).

Definition 1.5. Define a function D from $B_n(q)$ to $B_{n-1}(q)$ as follows. If $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, then $D(\mathbf{a}) = (b_1, b_2, \ldots, b_{n-1})$, where $b_i = a_{i+1} - a_i \bmod q$ for $i = 1$ to $n - 1$.

We extend the notation to allow the Lempel morphism D to be applied to periodic sequences in the natural way, as we now describe. That is, D is the map from the set of periodic sequences to itself defined by

$$D((s_i)) = (t_i) : t_j = s_{j+1} - s_j.$$

The image of a sequence of period m will clearly have period dividing m. In the usual way we can define D^{-1} to be the *pre-image* of D, i.e. if S is a periodic sequence than $D^{-1}(S)$ is the set of all sequences T with the property that $D(T) = S$.

The *weight* $w(S)$ of a sequence S is the weight of the ring sequence corresponding to S (that is the sum of the terms $s_0, \ldots, s_{m-1}$ treating s_i as an integer in the range $[0, q-1]$). Similarly we write $w_q(S)$ for $w(S) \bmod q$. The notion of weight is key to the rest of the paper since, when applied to an n-window sequence of period m and weight coprime to q, its pre-image under D^{-1} consists of q sequences of period qm whose ring sequences are cyclic shifts of each other. This enables us to identify a unique element of this set to be the *inverse*, e.g. the one starting with a zero.

1.3 Related work

This paper builds on the work of Alhakim et al. [2], in which recursive methods of construction for non-binary orientable sequences are described. Alhakim et al. described a range of methods of recursively generating orientable sequences using sequences with special properties, notably negative orientable and special orientable sequences. However, general methods for providing 'starter' sequences for these constructions were not provided, and this paper is aimed at addressing this.

In a recent paper [5], Gabrić and Sawada showed how to construct non-binary orientable sequences of asymptotically maximal period. Their approach involves applying the inverse Lempel Homomorphism to an orientable sequence and then demonstrating ways to join together the multiple sequences that result. In parallel work, Mitchell and Wild [9] showed how to construct orientable sequences using a rather different approach, namely first constructing negative orientable sequences and then applying certain methods of Alhakim et al. to construct larger period orientable sequences. This paper follows a similar path, except that we show how to construct special orientable sequences, and then use these in other methods of Alhakim et al. to construct larger period orientable sequences.

2 A simple period bound

By definition it follows automatically that the period of an $\mathcal{SOS}_q(n)$ is bounded above by the bounds on the period of an orientable sequence and that of a negative orientable sequence established in [2, Theorem 4.11] and [9, Theorem 3.10]. We next give a bound on the period of an $\mathcal{SOS}_q(n)$ which is of the same order as these general bounds.

Theorem 2.1. *Suppose S is an $\mathcal{SOS}_q(n)$. Then the period of S is at most:*

$$\begin{array}{ll} \dfrac{q^n - q^{(n+1)/2} - q^{(n-1)/2} + 1}{2} & \text{if } q \text{ and } n \text{ are both odd;} \\ \dfrac{q^n - 2q^{n/2} + 1}{2} & \text{if } q \text{ is odd and } n \text{ is even;} \\ \dfrac{q^n - q^{(n+1)/2} - 2q^{(n-1)/2} + 2^{(n+3)/2} - 2^{(n+1)/2}}{2} & \text{if } q \text{ is even and } n \text{ is odd;} \\ \dfrac{q^n - 2q^{n/2} + 2^{(n+2)/2} - 2^{n/2}}{2} & \text{if } q \text{ and } n \text{ are both even.} \end{array}$$

Proof. First observe that if an n-tuple $\mathbf{s}$ satisfies $\mathbf{s} = \mathbf{s}^R$ or $\mathbf{s} = -\mathbf{s}^R$ then it cannot occur in S since S is both orientable and negative orientable. Hence, since at most one of $\mathbf{s}$ and $\mathbf{s}^R$ can occur in S, the period of S is at most half the number of q-ary n-tuples s such that $\mathbf{s} \neq \mathbf{s}^R$ and $\mathbf{s} \neq -\mathbf{s}^R$. We examine the four cases separately. Note that the q odd cases are simpler, since when q is odd there is only one n-tuple satisfying $\mathbf{s} = -\mathbf{s}$, namely the all-zero n-tuple.

- Suppose q and n are both odd. Then there is one n-tuple $\mathbf{s}$ with $\mathbf{s} = -\mathbf{s} = \mathbf{s}^R = -\mathbf{s}^R$; $q^{(n+1)/2} - 1$ tuples with $\mathbf{s} = \mathbf{s}^R \neq -\mathbf{s} = -\mathbf{s}^R$; $q^{(n-1)/2} - 1$ with $\mathbf{s} = -\mathbf{s}^R \neq -\mathbf{s} = \mathbf{s}^R$; and hence there are h with $\mathbf{s}, -\mathbf{s}, \mathbf{s}^R, -\mathbf{s}^R$ all distinct, where $h = q^n - q^{(n+1)/2} - q^{(n-1)/2} + 1$. The bound is $h/2$, and the result follows.

- Suppose q is odd and n is even. Then there is one n-tuple $\mathbf{s}$ with $\mathbf{s} = -\mathbf{s} = \mathbf{s}^R = -\mathbf{s}^R$; $q^{n/2} - 1$ with $\mathbf{s} = \mathbf{s}^R \neq -\mathbf{s} = -\mathbf{s}^R$; $q^{n/2} - 1$ with $\mathbf{s} = -\mathbf{s}^R \neq -\mathbf{s} = \mathbf{s}^R$; and hence $h = q^n - 2q^{n/2} + 1$ with $\mathbf{s}, -\mathbf{s}, \mathbf{s}^R, -\mathbf{s}^R$ all distinct. The bound is $h/2$, and the result follows.

- Suppose q is even and n is odd. Then there are $2^{(n+1)/2}$ n-tuples $\mathbf{s}$ with $\mathbf{s} = -\mathbf{s} = \mathbf{s}^R = -\mathbf{s}^R$; $N = 2^n - 2^{(n+1)/2}$ n-tuples with $\mathbf{s} = -\mathbf{s} \neq \mathbf{s}^R = -\mathbf{s}^R$; $q^{(n+1)/2} - 2^{(n+1)/2}$ with $\mathbf{s} = \mathbf{s}^R \neq -\mathbf{s} = -\mathbf{s}^R$; $2q^{(n-1)/2} - 2^{(n+1)/2}$ with $\mathbf{s} = -\mathbf{s}^R \neq -\mathbf{s} = \mathbf{s}^R$; and hence $h = q^n - q^{(n+1)/2} - 2q^{(n-1)/2} + 2^{(n+3)/2} - 2^n$ with $\mathbf{s}, -\mathbf{s}, \mathbf{s}^R, -\mathbf{s}^R$ all distinct. The bound is $(N+h)/2$, and the result follows.

- Suppose q and n are both even. Then there are $2^{n/2}$ n-tuples $\mathbf{s}$ with $\mathbf{s} = -\mathbf{s} = \mathbf{s}^R = -\mathbf{s}^R$; $N = 2^n - 2^{n/2}$ n-tuples with $\mathbf{s} = -\mathbf{s} \neq \mathbf{s}^R = -\mathbf{s}^R$; $q^{n/2} - 2^{n/2}$ with $\mathbf{s} = \mathbf{s}^R \neq -\mathbf{s} = -\mathbf{s}^R$; $q^{n/2} - 2^{n/2}$ with $\mathbf{s} = -\mathbf{s}^R \neq -\mathbf{s} = \mathbf{s}^R$; and hence $h = q^n - 2q^{n/2} + 2^{(n+2)/2} - 2^n$ with $\mathbf{s}, -\mathbf{s}, \mathbf{s}^R, -\mathbf{s}^R$ all distinct. The bound is $(N+h)/2$, and the result follows.

□

3 Constructing special orientable sequences

3.1 A simple construction

We first show how to construct an $\mathcal{SOS}_q(n)$ with period about one quarter the bound given by Theorem 2.1 for every odd $q \geq 5$ when $n = 2$.

Construction 3.1. Let q, q' be integers with $q' > q > 1$. For $x \in \mathbb{Z}_q$ we write $\underline{x}$ for the non-negative integer in $\{0, 1, \ldots, q-1\}$ belonging to the residue class x, and x' for the residue class of $\mathbb{Z}_{q'}$ that contains $\underline{x}$. Let $S = [s_0, \ldots, s_{m-1}]$ be an $\mathcal{OS}_q(n)$. Let $S' = [s'_0, \ldots, s'_{m-1}]$ be the sequence over $\mathbb{Z}_{q'}$ obtained from S in the obvious notational way.

Theorem 3.1. *If S is an $\mathcal{OS}_q(n)$, $q' \geq 2q - 1$ and S' is obtained from S using Construction 3.1, then S' is an $\mathcal{SOS}_{q'}(n)$.*

Proof. First observe that if x is a non-zero term of S' then $-x \neq y$ for any term y of S'.

Suppose $0 \leq i, j < m$. We need to establish three properties.

- *S' is an n-window sequence.* Suppose $\mathbf{s}'_n(i) = \mathbf{s}'_n(j)$. Then $\mathbf{s}_n(i) = \mathbf{s}_n(j)$ and so $i \equiv j \pmod{m}$ (i.e. $i = j$).
- *S' is orientable.* Suppose $\mathbf{s}'_n(i) = \mathbf{s}'^R_n(j)$. Then $\mathbf{s}_n(i) = \mathbf{s}^R_n(j)$. This is impossible since S is an $\mathcal{OS}_q(n)$.
- *S' is negative orientable.* Finally, suppose $\mathbf{s}'_n(i) = -\mathbf{s}'^R_n(j)$. Then, by the observation above, $s'_i = s'_{i+1} = \cdots = s'_{i+n-1} = 0$ so that $s_i = s_{i+1} = \cdots = s_{i+n-1} = 0$, contradicting the assumption that S is an $\mathcal{OS}_q(n)$.

□

When $n = 2$, this allows us to give the following.

Corollary 3.2. *There exists an $\mathcal{SOS}_q(2)$ of period about one quarter of the maximum period given in Theorem 2.1) for all $q \geq 5$.*

Proof. From [9, Lemma 2.2] there exists an $\mathcal{OS}_q(2)$ with period either $q(q-1)/2$ (q odd) or $q(q-2)/2$ (q even) for every $q \geq 3$. From Construction 3.1, this implies the existence of an $\mathcal{SOS}_{2q-1}(2)$ and an $\mathcal{SOS}_{2q}(2)$ with period either $q(q-1)/2$ (q odd) or $q(q-2)/2$ (q even) for every $q \geq 3$. The result follows, since (by Theorem 2.1) the maximum period for an $\mathcal{SOS}_{2q-1}(2)$ is $(2q-2)^2/2$ and the maximum period for an $\mathcal{SOS}_{2q}(2)$ is $((2q-1)^2+1)/2$. □

3.2 A second construction

We next modify the method given immediately above to double the period and so enable the construction of special orientable sequences with period approximately half the maximum when $n = 2$. We do so by means of a general result regarding the relationship between a sequence and its negative.

Following Alhakim et al. [2] we make the following definition.

Definition 3.1. Suppose $S = (s_i)$ and $T = (t_i)$ are n-window sequences. They are said to be *special-orientable-disjoint (s-disjoint)* if:

1. they are n-tuple disjoint, i.e. $\mathbf{s}_n(i) \neq \mathbf{t}_n(j)$ for any i, j;
2. they are orientable disjoint (o-disjoint), i.e. $\mathbf{s}_n(i) \neq \mathbf{t}_n(j)^R$ for any i, j; and
3. they are negative orientable disjoint (n-disjoint), i.e. $\mathbf{s}_n(i) \neq -\mathbf{t}_n(j)^R$ for any i, j.

We can now state the following result.

Theorem 3.3. *Suppose S is an $\mathcal{SOS}_q(n)$ with the property that, for any n-tuple $\mathbf{s}$, at most one of $\mathbf{s}$ and $-\mathbf{s}$ is contained in S. Then S and $-S$ are s-disjoint.*

Proof. S and $-S$ are clearly n-tuple disjoint since we assumed that at most one of $\mathbf{s}$ and $-\mathbf{s}$ is contained in S for any $\mathbf{s}$. Now $\mathbf{s}_n(i) \neq -\mathbf{s}_n(j)^R$ for all i, j since S is an $\mathcal{NOS}_q(n)$, and hence S and $-S$ are o-disjoint. Finally, $\mathbf{s}_n(i) \neq -(-\mathbf{s}_n(j)^R) = \mathbf{s}_n(j)^R$ for all i, j since S is an $\mathcal{OS}_q(n)$, and hence S and $-S$ are n-disjoint. □

Remark 3.1. It follows immediately from Theorem 3.3 that if S is an $\mathcal{SOS}_q(n)$ of period m with the property that, for any n-tuple $\mathbf{s}$, at most one of $\mathbf{s}$ and $-\mathbf{s}$ is contained in S, and if in addition S and $-S$ share an $(n-1)$-tuple, then S and $-S$ can be joined to form an $\mathcal{SOS}_q(n)$ with period $2m$. This follows since, when concatenating s-disjoint sequences, the only possible problem arises for n-tuples that 'cross the join', and by joining them on a common $n-1$ tuple we can avoid creating any new n-tuples.

Next observe that any sequence S obtained from Construction 3.1 has the property that, for any n-tuple $\mathbf{s}$, at most one of $\mathbf{s}$ and $-\mathbf{s}$ is contained in S. This immediately motivates the following construction.

Construction 3.2. Let q, q' be integers with $q' \geq 2q - 1 > 2$. For $x \in \mathbb{Z}_q$ we write $\underline{x}$ for the non-negative integer in $\{0, 1, \ldots, q-1\}$ belonging to the residue class x and x' for the residue class of $\mathbb{Z}_{q'}$ that contains $\underline{x}$.

Let $S = [s_0, s_1, \ldots, s_{m-1}]$ be an $\mathcal{OS}_q(n)$. Let $S' = [s'_0, s'_1, \ldots, s'_{m-1}]$ be the sequence over $\mathbb{Z}_{q'}$ obtained from S in the obvious notational way. Let $S'' = [s''_0, s''_1, \ldots, s''_{2m-1}]$ be the periodic sequence whose ring sequence is the concatenation of the ring sequences of S' and $-S'$.

We next introduce some notation. Let q, q' be integers with $q' \geq 2q - 1 > 2$. As in Construction 3.2, given $x \in \mathbb{Z}_q$, we write $\underline{x}$ for the integer in $\{0, 1, \ldots, q-1\}$ belonging to the residue class x, and x' for the residue class of $\mathbb{Z}_{q'}$ that contains $\underline{x}$. Similarly, for $y \in \mathbb{Z}_{q'}$ we write $\underline{y}$ for the integer in $\{0, 1, \ldots, q'-1\}$ belonging to the residue class y. Let $E_{q,q'} : \mathbf{Z}_q \to \mathbf{Z}_{q'}$ be the mapping given by $E_{q,q'}(x) = x'$ for all $x \in \mathbf{Z}_q$.

Let $M_{q,q'} : \mathbf{Z}_{q'} \to \mathbf{Z}_q$ be the mapping given by

$$M_{q,q'}(y) = \begin{cases} x & \text{when } 0 \leq \underline{y} = \underline{x} \leq q-1 \text{ (so that } x' = y\text{),} \\ 0 & \text{when } q \leq \underline{y} \leq q' - q, \text{ and} \\ x & \text{when } q' - q + 1 \leq \underline{y} \leq q' - 1 \text{ and } \underline{x} = q' - \underline{y} \text{ (so that } x' = -y\text{).} \end{cases}$$

When q and q' are understood we simply write E and M for $E_{q,q'}$ and $M_{q,q'}$ respectively.

Note that it follows immediately from the definitions of E and M that $M(-y) = M(y)$ for all $y \in \mathbf{Z}_{q'}$, in particular $M(E(x)) = M(-E(x)) = x$ for all $x \in \mathbf{Z}_q$. We extend the application of E and M to n-tuples and sequences in the natural way, that is by applying them to each term. So, in Construction 3.5, $S' = E(S)$ and $M(\mathbf{s}''_n(i)) = \mathbf{s}_n(i)$, $M(\mathbf{s}''^R_n(i)) = \mathbf{s}^R_n(i)$ and $M(-s''_n(i)) = \mathbf{s}_n(i)$ for all i.

Theorem 3.4. *If S is an $\mathcal{OS}_q(n)$ and S'' is obtained from S using Construction 3.2 then S'' is an $\mathcal{SOS}_{q'}(n)$ with $w_{q'}(S'') = 0$.*

Proof. We establish three properties.

- S'' *is an n-window sequence.* Suppose $\mathbf{s}''_n(i) = \mathbf{s}''_n(j)$. Then $M(\mathbf{s}''_n(i)) = M(\mathbf{s}''_n(j))$, that is $\mathbf{s}_n(i) = \mathbf{s}_n(j)$ and so $i \equiv j \pmod{m}$ as S is an n-window sequence of period m. Since $\mathbf{s}''_n(i+m) = -\mathbf{s}''_n(i)$ and $\mathbf{s}_n(i)$ cannot have every term equal to 0, we deduce that $i \equiv j \pmod{2m}$.
- S'' *is orientable.* Suppose $\mathbf{s}''_n(i) = \mathbf{s}''^R_n(j)$. Then $M(\mathbf{s}''_n(i)) = M(\mathbf{s}''^R_n(j))$, that is $\mathbf{s}_n(i) = \mathbf{s}^R_n(j)$ which is impossible as S is an orientable sequence.
- S'' *is negative orientable.* Finally, suppose $\mathbf{s}''_n(i) = -\mathbf{s}''^R_n(j)$. Then $M(\mathbf{s}''_n(i)) = M(-\mathbf{s}''^R_n(j))$, that is $\mathbf{s}_n(i) = \mathbf{s}^R_n(j)$ which is impossible as S is an orientable sequence.

The result follows, observing that $w_{q'}(S'') = w_{q'}(S') + w_{q'}(-S') = w_{q'}(S') - w_{q'}(S') = 0$. □

The following simple example demonstrates Construction 3.2.

Example 3.1. First observe that $S = [01234\ 02413]$ is an $\mathcal{OS}_5(2)$ (obtained using Construction 5.3 of [2])[1].

If we put $q' = 9$, then $S' = [01234\ 02413]$ and

$$S'' = S'||(-S') = [01234\ 02413\ 08765\ 07586]$$

(where $||$ denotes sequence concatenation). It follows from Theorem 3.4 that S'' is an $\mathcal{SOS}_9(2)$.

We can also perform the same construction with $q' = 10$. In this case

$$S'' = S'||(-S') = [01234\ 02413\ 09876\ 08697]$$

and S'' is an $\mathcal{SOS}_{10}(2)$.

Corollary 3.5. *There exists an $\mathcal{SOS}_q(2)$, of period*

$$\begin{aligned} \frac{q(q-4)}{4} & \quad \textit{if} \quad q \equiv 0 \pmod 4, \\ \frac{(q+1)(q-1)}{4} & \quad \textit{if} \quad q \equiv 1 \pmod 4, \\ \frac{q(q-2)}{4} & \quad \textit{if} \quad q \equiv 2 \pmod 4, \\ \frac{(q+1)(q-3)}{4} & \quad \textit{if} \quad q \equiv 3 \pmod 4, \end{aligned}$$

for all $q \geq 5$.

Proof. Suppose $q \geq 5$. If $q \equiv 0 \pmod 4$ then $\frac{q}{2}$ is even and by [9, Lemma 2.2] there exists an $\mathcal{OS}_{\frac{q}{2}}(2)$ with period $\frac{q}{2}(\frac{q}{2} - 2)/2$; hence by Theorem 3.4 there exists an $SOS_q(2)$ of period $\frac{q(q-4)}{4}$.

If $q \equiv 1 \pmod 4$ then $\frac{q+1}{2}$ is odd and by [9, Lemma 2.2] there exists an $OS_{\frac{q+1}{2}}(2)$ with period $\frac{q+1}{2}(\frac{q+1}{2} - 1)/2$; hence by Theorem 3.4 there exists an $SOS_q(2)$ of period $\frac{(q+1)(q-1)}{4}$.

If $q \equiv 2 \pmod 4$ then $\frac{q}{2}$ is odd and by [9, Lemma 2.2] there exists an $\mathcal{OS}_{\frac{q}{2}}(2)$ with period $\frac{q}{2}(\frac{q}{2} - 1)/2$; hence by Theorem 3.4 there exists an $SOS_q(2)$ of period $\frac{q(q-2)}{4}$.

If $q \equiv 3 \pmod 4$ then $\frac{q+1}{2}$ is even and by [9, Lemma 2.2] there exists an $OS_{\frac{q+1}{2}}(2)$ with period $\frac{q+1}{2}(\frac{q+1}{2} - 2)/2$; hence by Theorem 3.4 there exists an $SOS_q(2)$ of period $\frac{(q+1)(q-3)}{4}$.

□

[1] Here and in other examples the spaces are included simply to make reading easier.

We remark that the period of these Special Orientable Sequences of order 2 is approximately half that of the maximum period given by Theorem 2.1.

3.3 Extending the construction

We now further modify the previous constructions, doubling the period again, to enable us to obtain special orientable sequences which have period of the same order as the bound of Theorem 2.1 when $n = 2$.

Construction 3.3. Let q, q' be integers with $q' \geq 2q > 3$. Let $S = [s_0, \ldots, s_{m-1}]$ be an $\mathcal{OS}_q(n)$ and let $S' = [s'_0, \ldots, s'_{m-1}]$ be the $\mathcal{OS}_{q'}(n)$ constructed as in Construction 3.2. Let $T = [t_0, t_1, \ldots, t_{m-1}]$ be the sequence over $\mathbb{Z}_{q'}$ such that $t_i = (-1)^{i+m-1}s'_i$ for $i = 0, \ldots, m-1$ unless $s'_i = 0$ in which case $t_i = (-1)^{i+m-1}q$.

Lemma 3.6. *The sequence T of Construction 3.3 is an $\mathcal{SOS}_n(q')$.*

Proof. We establish three properties.

- *T is an n-window sequence.* Suppose $\mathbf{t}_n(i) = \mathbf{t}_n(j)$. Then $M(\mathbf{t}_n(i)) = M(\mathbf{t}_n(j))$, so that $\mathbf{s}_n(i) = \mathbf{s}_n(j)$ since $M(q) = M(-q) = 0$. Hence $i = j \pmod m$ as S is an n-window sequence of period m.
- *T is orientable.* Suppose $\mathbf{t}_n(i) = \mathbf{t}^R_n(j)$. Then $M(\mathbf{t}_n(i)) = M(\mathbf{t}^R_n(j))$, so that $\mathbf{s}_n(i) = \mathbf{s}^R_n(j)$, which is impossible as S is an orientable sequence.
- *T is negative orientable.* Finally, suppose $\mathbf{t}_n(i) = -\mathbf{t}^R_n(j)$. Then $M(\mathbf{t}_n(i)) = M(-\mathbf{t}^R_n(j))$, that is $\mathbf{s}_n(i) = \mathbf{s}^R_n(j)$, which is impossible as S is an orientable sequence.

The result follows. □

An example of Construction 3.3 follows.

Example 3.2. As in Example 3.1, let $S = [01234\ 02413]$ be the $\mathcal{OS}_5(2)$ obtained using Construction 5.3 of [2].

If we put $q' = 10$, then, as $m = 10$, $(-1)^{m-1} = -1$ and

$$T = [51836\ 58493].$$

It follows from Lemma 3.6 that T is an $\mathcal{SOS}_{10}(2)$.

We next show that we can adjoin $-T$ to T to obtain an $\mathcal{SOS}$ with twice the period, just as was the case with S'.

Construction 3.4. Let q, q', n be integers with $q' \geq 2q > 3$ and $n > 1$. Let $S = [s_0, \ldots, s_{m-1}]$ be an $\mathcal{OS}_q(n)$. Let $T = [t_0, t_1, \ldots, t_{m-1}]$ be as in Construction 3.3. Let T' be the sequence whose ring sequence is the concatenation of the ring sequences of T and $-T$.

Theorem 3.7. *If S is an $\mathcal{OS}_q(n)$ and T' is obtained from S using Construction 3.4 then T' is an $\mathcal{SOS}_{q'}(n)$ with $w_{q'}(T') = 0$.*

Proof. We establish three properties.

- *T' is an n-window sequence.* Suppose $\mathbf{t}'_n(i) = \mathbf{t}'_n(j)$. Then $M(\mathbf{t}'_n(i)) = M(\mathbf{t}'_n(j))$, so that $\mathbf{s}_n(i) = \mathbf{s}_n(j)$ since $M(q) = M(-q) = 0$. Hence $i \equiv j \pmod{m}$ as S is an n-window sequence of period m. Since $\mathbf{t}'_n(i+m) = -\mathbf{t}'_n(i) \neq \mathbf{t}'_n(i)$, as $\mathbf{s}_n(i)$ is not the all 0 tuple, we must have $j = i \pmod{2m}$.
- *T' is orientable.* Suppose $\mathbf{t}'_n(i) = \mathbf{t}'^R_n(j)$. Then $M(\mathbf{t}'_n(i)) = M(\mathbf{t}'^R_n(j))$, so that $\mathbf{s}_n(i) = \mathbf{s}^R_n(j)$, which is impossible as S is an orientable sequence.
- *T' is negative orientable.* Finally, suppose $\mathbf{t}'_n(i) = -\mathbf{t}'^R_n(j)$. Then $M(\mathbf{t}'_n(i)) = M(-\mathbf{t}'^R_n(j))$, that is $\mathbf{s}_n(i) = \mathbf{s}^R_n(j)$, which is impossible as S is an orientable sequence.

The result follows, observing that $w_{q'}(T') = w_{q'}(T) + w_{q'}(-T) = w_{q'}(T) - w_{q'}(T) = 0$. □

We extend our previous example to give an example of Construction 3.4.

Example 3.3. As in Examples 3.1 and 3.2, let $S = [01234\ 02413]$ be the $\mathcal{OS}_5(2)$ obtained using Construction 5.3 of [2].

If we put $q' = 10$, then, as in Example 3.2

$$T = [51836\ 58493].$$

We then have that

$$T' = T||(-T) = [51836\ 58493\ 59274\ 52617].$$

It follows from Theorem 3.7 that T' is an $\mathcal{SOS}_{10}(2)$.

We complete the extended construction by combining the sequence T' generated using Construction 3.4 with the sequence S'' generated using Construction 3.2.

Corollary 3.8. *Let q, q' be integers with $q' \geq 2q+1 > 4$. Let $S = [s_0, \ldots, s_{m-1}]$ be an $\mathcal{OS}_q(n)$ of period m with $s_0 = 0$. Let S'' be obtained from S as in Construction 3.2 and let T' be obtained from S as in Construction 3.4. Then $S^{''}$ and T' are s-disjoint.*

Proof. We consider three cases.

- Suppose $\mathbf{s}''_n(i) = \mathbf{t}'_n(j)$. Then $M(\mathbf{s}''_n(i)) = M(\mathbf{t}'_n(j))$, so that $\mathbf{s}_n(i) = \mathbf{s}_n(j)$ since $M(q) = M(-q) = 0$. Hence $i \equiv j \pmod{m}$ as S is an n-window sequence of period m. Suppose $0 \leq i \leq 2m-1$. Then for some k with $0 \leq k \leq n-1$, we have that $s''_{i+\ell}$ for $\ell = 0, \ldots, k$ all lie in $\{0, 1, \ldots, q-1\}$ or in $\{0, -1, \ldots, -(q-1)\}$ and $s''_{i+\ell}$ for $\ell = k+1, \ldots, n-1$ all lie in $\{0, -1, \ldots, -(q-1)\}$ or in $\{0, 1, \ldots, (q-1)\}$ respectively while the terms of $\mathbf{t}'_n(j)$ alternate between the two sets $\{1, \ldots, q\}$ and $\{-1, \ldots, -q\}$ unless m is even and $\mathbf{t}'_n(j)$ contains $t'_{\alpha m-1}$ and $t'_{\alpha m}$ for some $\alpha \geq 1$. Since $n > 1$ and $s''_{\alpha m} = 0$ so that $t'_{\alpha m} = \pm q$ this is not possible. It follows that $S^{''}$ and T' are n-window disjoint.
- Suppose $\mathbf{s}''_n(i) = \mathbf{t}'^R_n(j)$. Then $M(\mathbf{s}''_n(i)) = M(\mathbf{t}'^R_n(j))$, so that $\mathbf{s}_n(i) = \mathbf{s}^R_n(j)$ since $M(q) = M(-q) = 0$. Hence $i \equiv j \pmod{m}$ as S is an orientable sequence of period m. Now a similar argument as above about where the terms of $\mathbf{s}''_n(i)$ and $\mathbf{t}'^R_n(j)$ lie shows that the supposition is impossible and it follows that S'' and T' are o-disjoint.
- Finally, suppose $\mathbf{s}''_n(i) = -\mathbf{t}'^R_n(j)$. Then $M(\mathbf{s}''_n(i)) = M(-\mathbf{t}'^R_n(j))$, so that $\mathbf{s}_n(i) = \mathbf{s}^R_n(j)$ since $M(q) = M(-q) = 0$. Hence $i \equiv j \pmod{m}$ as S is an orientable sequence of period m. Now a similar argument as above about where the terms of $\mathbf{s}''_n(i)$ and $\mathbf{t}'^R_n(j)$ lie shows that the supposition is impossible and it follows that S'' and T' are n-disjoint.

The result follows. □

Corollary 3.9. *Suppose q, q', n are integers satisfying $q' \geq 2q+1$, $q > 1$ and $n > 1$. If S is an $\mathcal{OS}_q(n)$ of period m with $s_0 = 0$, and S'' and T' are obtained from S using Constructions 3.2 and 3.4, then the ring sequences of S'' and T' may be concatenated to obtain the ring sequence of an $\mathcal{SOS}_{q'}(n)$ U of period $4m$, where $w_{q'}(U) = 0$.*

Proof. As $S^{''}$ and T' are s-disjoint $\mathcal{SOS}_{q'}(n)$ we need only check that the n-tuples $\mathbf{u}_n(i)$, $i = 2m-n+1, \ldots, 2m-1$ and $i = 4m-n+1, \ldots, 4m-1$ do not appear as $\mathbf{u}_n(j)$ for any $j \not\equiv i \pmod{4m}$, nor as $\mathbf{u}^R_n(j)$ for any j, nor as $-\mathbf{u}^R_n(j)$ for any j. Suppose $\mathbf{u}_n(i)$, with $i \in \{2m-n+1, \ldots, 2m-1\}$ or with $i \in \{4m-n+1, \ldots, 4m-1\}$ equals $\mathbf{u}_n(j)$ for some j. Then $M(\mathbf{u}_n(i)) = M(\mathbf{u}_n(j))$ are n-tuples of S so that $j \equiv i \pmod{m}$. We now need only check that for $\ell = 1, \ldots n-1$ the four n-tuples $\mathbf{u}_{m-\ell}, \mathbf{u}_{2m-\ell}, \mathbf{u}_{3m-\ell}, \mathbf{u}_{4m-\ell}$

are distinct. This follows if the four 2-tuples (u_{m-1}, u_m), (u_{2m-1}, u_{2m}), (u_{3m-1}, u_{3m}), (u_{4m-1}, u_{4m}) are distinct. That is $(s'_{m-1}, 0)$, $(-s'_{m-1}, (-1)^{m-1}q)$, $(t_{m-1}, (-1)^m q)$, $(-t_{m-1}, 0)$ are distinct. This is easily checked, knowing that $t_{m-1} = (-1)^{2m-2}s'_{m-1} \neq -s'_{m-1}$ unless $s'_{m-1} = 0$ in which case $t_{m-1} = (-1)^{2m-2}q$. It follows that $j = i \pmod{4m}$ and U is an n-window sequence.

Similar arguments as before, using the mapping M, show that $\mathbf{u}_n(i)$ does not equal $\mathbf{u}_n^R(j)$ or $-\mathbf{u}_n^R(j)$ for any j, so U is both orientable and negative orientable. Thus U is an $\mathcal{SOS}_{q'}(n)$. The result follows, observing that $w_{q'}(U) = w_{q'}(S'') + w_{q'}(-S'') + w_{q'}(T') + w_{q'}(-T') = w_{q'}(S'') - w_{q'}(-S'') + w_{q'}(T') - w_{q'}(-T') = 0$. □

A simple example of Corollary 3.9 is as follows.

Example 3.4. Suppose $q = 5$, $q' = 11$ and $n = 2$. As previously, we build upon the $\mathcal{OS}_5(2)$ with ring sequence $S = [01234\ 02413]$. Analogously to the second part of Example 3.1 we have

$$S'' = [0, 1, 2, 3, 4,\ 0, 2, 4, 1, 3,\ 0, 10, 9, 8, 7,\ 0, 9, 7, 10, 8].$$

Analogously to Example 3.3 we have

$$T' = [6, 1, 9, 3, 7,\ 5, 9, 4, 10, 3,\ 5, 10, 2, 8, 4,\ 6, 2, 7, 1, 8].$$

We simply concatenate them to obtain

$$\begin{aligned} U =& [0, 1, 2, 3, 4,\ 0, 2, 4, 1, 3,\ 0, 10, 9, 8, 7,\ 0, 9, 7, 10, 8, \\ & 6, 1, 9, 3, 7,\ 5, 9, 4, 10, 3,\ 5, 10, 2, 8, 4,\ 6, 2, 7, 1, 8] \end{aligned}$$

which by Corollary 3.9 is an $\mathcal{SOS}_{11}(2)$.

Corollary 3.10. *There exists an $\mathcal{SOS}_q(2)$ of period:*

$$\begin{aligned} &\frac{(q-2)(q-4)}{2} \quad \textit{if } q \equiv 0 \pmod 4, \\ &\frac{(q-1)(q-5)}{2} \quad \textit{if } q \equiv 1 \pmod 4, \\ &\frac{(q-2)(q-6)}{2} \quad \textit{if } q \equiv 2 \pmod 4, \\ &\frac{(q-1)(q-3)}{2} \quad \textit{if } q \equiv 3 \pmod 4, \end{aligned}$$

for all $q \geq 6$.

Proof. Suppose $q \geq 6$.

- If $q \equiv 0 \pmod 4$ then $\frac{q-2}{2}$ is odd, and by [9, Lemma 2.2] there exists an $\mathcal{OS}_{\frac{q-2}{2}}(2)$ with period $\frac{q-2}{2}(\frac{q-2}{2}-1)/2$. So by Corollary 3.9 there exists an $\mathcal{SOS}_q(2)$ of period $4\frac{q-2}{2}(\frac{q-2}{2}-1)/2 = \frac{(q-2)(q-4)}{2}$.
- If $q \equiv 1 \pmod 4$ then $\frac{q-1}{2}$ is even, and by [9, Lemma 2.2] there exists an $\mathcal{OS}_{\frac{q-1}{2}}(2)$ with period $\frac{q-1}{2}(\frac{q-1}{2}-2)/2$. So by Corollary 3.9 there exists an $\mathcal{SOS}_q(2)$ of period $4\frac{q-1}{2}(\frac{q-1}{2}-2)/2 = \frac{(q-1)(q-5)}{2}$.
- If $q \equiv 2 \pmod 4$ then $\frac{q-2}{2}$ is even, and by [9, Lemma 2.2] there exists an $\mathcal{OS}_{\frac{q-2}{2}}(2)$ with period $\frac{q-2}{2}(\frac{q-2}{2}-2)/2$. So by Corollary 3.9 there exists an $\mathcal{SOS}_q(2)$ of period $4\frac{q-2}{2}(\frac{q-2}{2}-2)/2 = \frac{(q-2)(q-6)}{2}$.
- If $q \equiv 3 \pmod 4$ then $\frac{q-1}{2}$ is odd, and by [9, Lemma 2.2] there exists an $\mathcal{OS}_{\frac{q-1}{2}}(2)$ with period $\frac{q-1}{2}(\frac{q-1}{2}-1)/2$. So by Corollary 3.9 there exists an $\mathcal{SOS}_q(2)$ of period $4\frac{q-1}{2}(\frac{q-1}{2}-1)/2 = \frac{(q-1)(q-3)}{2}$.

□

Observe that all the constructed sequences have q'-ary weight zero.

3.4 Adjusting the weight

Our main objective in giving the above constructions is to provide 'starter sequences' for certain constructions of Alhakim et al. [2]. However, all the sequences constructed here have weight zero; in particular, the sequence U obtained in Corollary 3.9 satisfies $w_{q'}(U) = 0$. We would ideally like to construct sequences U^* such that $w_{q'}(U^*)$ is coprime to q'. Therefore we next describe how to modify the sequences U of Corollary 3.9 in the case $n = 2$ to obtain sequences with precisely this property.

We first need the following simple result.

Lemma 3.11. *Suppose $q > 4$. Then, for any distinct x, y, z in $\mathbb{Z}_q$, there exists an $\mathcal{OS}_q(2)$ of maximal period, i.e. of period $q(q-1)/2$ (q odd) or $q(q-2)/2$ (q even), such that its ring sequence has the form $[xyzx\ldots]$. Moreover, if $x, y, z \neq 0$ then there exists an $\mathcal{OS}_q(2)$ of maximal period such that its ring sequence has the form $[0xyzx\ldots]$.*

Proof. If q is odd then, from Lemma 2.2 of [9], there exists an $\mathcal{OS}_q(2)$ of period $q(q-1)/2$ corresponding to an Eulerian circuit in K_q, the complete graph on q vertices. Every vertex has degree $q-1$, which is at least 4 since $q \geq 5$, and hence there exists an Eulerian circuit in K_q starting with the edges (x, y), (y, z), (z, x), and, should $x, y, z \neq 0$, an Eulerian circuit starting with the edges $(0, x), (x, y)$, (y, z), (z, x). The result follows.

If q is even, then (again from Lemma 2.2 of [9]), there exists an $\mathcal{OS}_q(2)$ of period $q(q-2)/2$ corresponding to an Eulerian circuit in K_q^*, where K_q^* is K_q with an arbitrary one-factor removed. Since $q \geq 6$, it is simple to choose a one-factor which avoids the edges (x, y), (y, z), and (z, x) or, should $x, y, z \neq 0$, the edges $(0, x)$, (x, y), (y, z), and (z, x); moreover the vertices in K_q^* will have degree at least 4. As a result there will exist an Eulerian circuit in K_q^* starting with the edges (x, y), (y, z), (z, x) and, should $x, y, z \neq 0$, with edges $(0, x)$, (x, y), (y, z), and (z, x). The result follows. □

Construction 3.5. Suppose $q > 2$ and $q' = 2q + 1$ or $q' = 2q + 2$. If $q \geq 5$ and $q' = 2q + 1$, set $x = 2$, $y = q - 2$ and $z = q - 1$ (and so $x + y + z = 2q - 1$). If $q \geq 7$ and $q' = 2q + 2$, set $x = 4$, $y = q - 2$ and $z = q - 1$ (and so $x + y + z = 2q + 1$). Otherwise set x, y and z according to Table 1.

Table 1: Choosing x, y and z

q	q'	x	y	z	$x+y+z$
5	12	0	1	4	5
6	14	0	1	2	3

First observe that, in all cases x, y and z are distinct and $y, z \neq 0$. By inspection it also holds that $x+y+z$ is coprime to q' for all possible choices of q and q'.

Suppose S is an $\mathcal{OS}_q(2)$ of maximal period m (i.e. of period $q(q-1)/2$ (q odd) or $q(q-2)/2$ (q even)), such that its ring sequence has the form $[xyzx\ldots]$ or, should $x, y, z \neq 0$, the form $[0xyzx\ldots]$, which exists from Lemma 3.11. Construct U from S using the method of Corollary 3.9. Observe that, from the method of construction, the ring sequence for U has the form $[xyzx\ldots]$ or, should $x \neq 0$, the form $[0xyzx\ldots]$. Finally, construct U^* from U by deleting the cycle $[xyz]$ from its ring sequence.

Theorem 3.12. *Suppose $q > 4$ and $q' = 2q + 1$ or $q' = 2q + 2$. If U^* is constructed according to the method of Construction 3.5 then it is an $\mathcal{SOS}_{q'}(2)$ of period $2q(q-1) - 3$ (q odd) or $2q(q-2) - 3$ (q even) where in every case $w_{q'}(U^*)$ is coprime to q'.*

Proof. By Corollary 3.9, the sequence U is an $\mathcal{SOS}_{q'}(2)$ of period $2q(q-1)$ (q odd) or $2q(q-1)$ (q odd) where $w_{q'}(U) = 0$. The result now follows immediately by observing that constructing U^* from U does not add any new 2-tuples, that $w_{q'}(U^*) = q' - (x+y+z)$, and, as noted above, $x+y+z$ is coprime to q'. □

The following brief example shows the operation of this construction.

Example 3.5. Suppose $q = 5$ and $q' = 11$. In this case $x = 2$, $y = 3$ and $z = 4$, and so we need an $\mathcal{OS}_2(q)$ of maximal period with ring sequence of the form [02342 . . .]. An example of such a sequence is S = [02342 10314]. Then S = [02342 10314] and

$$S'' = S'||(-S') = [0,2,3,4,2,\ 1,0,3,1,4,\ 0,9,8,7,9,\ 10,0,8,10,7]$$

(where || denotes sequence concatenation). It follows from Theorem 3.4 that S'' is an $\mathcal{SOS}_{11}(2)$. We next have

$$T = [6,2,8,4,9,\ 1,6,3,10,4],$$

where, from Lemma 3.6, T is an $\mathcal{SOS}_{11}(2)$. Then

$$T' = T||(-T) = [6,2,8,4,9,\ 1,6,3,10,4,\ 5,9,3,7,2,\ 10,5,8,1,7].$$

We next concatenate S'' and T' to obtain

$$\begin{aligned} U =&[0,2,3,4,2,\ 1,0,3,1,4,\ 0,9,8,7,9,\ 10,0,8,10,7\\ &6,2,8,4,9,\ 1,6,3,10,4,\ 5,9,3,7,2,\ 10,5,8,1,7]. \end{aligned}$$

which by Corollary 3.9 is an $\mathcal{SOS}_{11}(2)$. Finally we simply delete the cycle [234] from U to obtain

$$\begin{aligned} U^* =&[0,2,\ 1,0,3,1,4,\ 0,9,8,7,9,\ 10,0,8,10,7\\ &6,2,8,4,9,\ 1,6,3,10,4,\ 5,9,3,7,2,\ 10,5,8,1,7]. \end{aligned}$$

which is an $\mathcal{SOS}_{11}(2)$ of period 37 with $w_{11}(U^*) = 2$.

We also have the following simple corollary, which follows immediately from Corollary 3.10.

Corollary 3.13. *There exists an $\mathcal{SOS}_q(2)$ U^* of period:*

$$\begin{aligned} &\frac{(q-2)(q-4)}{2} - 3 \quad \textit{if} \quad q \equiv 0 \pmod 4,\\ &\frac{(q-1)(q-5)}{2} - 3 \quad \textit{if} \quad q \equiv 1 \pmod 4,\\ &\frac{(q-2)(q-6)}{2} - 3 \quad \textit{if} \quad q \equiv 2 \pmod 4,\\ &\frac{(q-1)(q-3)}{2} - 3 \quad \textit{if} \quad q \equiv 3 \pmod 4, \end{aligned}$$

for all $q \geq 11$, where $w_q(U^*)$ is coprime to q.

4 Good special orientable sequences

We next consider how to construct *good* special orientable sequences, given that this additional property enables us to apply certain recursive constructions from Alhakim at al. [2]. We first need the following.

Definition 4.1 ([2])**.** An orientable (respectively negative orientable) sequence with the property that any run of 0 has length at most $n-2$ is said to be *good*.

4.1 An initial observation

We immediately have the following, although the sequences have period only of the order of half the bound of Theorem 2.1.

Theorem 4.1. *There exists a good $\mathcal{SOS}_q(2)$, of period*

$$\begin{aligned} \frac{q(q-4)}{4} & \quad \text{if} \quad q \equiv 0 \pmod 4, \\ \frac{(q+1)(q-1)}{4} & \quad \text{if} \quad q \equiv 1 \pmod 4, \\ \frac{q(q-2)}{4} & \quad \text{if} \quad q \equiv 2 \pmod 4, \\ \frac{(q+1)(q-3)}{4} & \quad \text{if} \quad q \equiv 3 \pmod 4, \end{aligned}$$

for all $q \geq 5$.

Proof. The $OS_{q'}(n)$ T' of Theorem 3.7 is good by construction. The result follows using the same argument as in Corollary 3.5. □

In the remainder of this section we show how we can do considerably better than this.

4.2 A simple modification

A simple method of constructing a good special orientable sequence arises from the observation that an $\mathcal{SOS}_q(n)$ that possesses no zeros is automatically a good $\mathcal{SOS}_q(n)$. With this is mind we modify the sequences U of Corollary 3.9. Note that such a sequence U will always contain an even number of zeros, since in the sequences S' and $-S'$ that are concatenated to construct U, every zero in S' will give rise to a zero in $-S'$.

Construction 4.1. Suppose q, q', n are integers satisfying $q' \geq 2q+2$, $q > 1$ and $n > 1$. Suppose U is an $\mathcal{SOS}_{q'}(n)$ constructed according to

Corollary 3.9. Then let U' be derived from U by replacing half of the zeros with $q+1$ and the other half with $q'-q-1$.

Theorem 4.2. *Suppose q, q', n are integers satisfying $q' \geq 2q+2$, $q > 1$ and $n > 1$. Suppose U is an $\mathcal{SOS}_{q'}(n)$ constructed according to Corollary 3.9. If U' is derived from U using Construction 4.1, then U' is a good $\mathcal{SOS}_{q'}(n)$ of the same period as U, and $w_{q'}(U') = 0$.*

Proof. If we can show that U does not contain any occurrences of $q+1$ or $q'-q-1$ then the main result will follow immediately. Now U is constructed by concatenating sequences S'' and T', obtained using Constructions 3.2 and 3.4, so we next examine these two sequences.

S'' is obtained by concatenating sequences S' and $-S'$, where S is an $\mathcal{OS}_q(n)$. Now S' contains only elements between 0 and $q-1$ inclusive, and $-S'$ contains only 0 or elements between $q'-q+1$ and $q'-1$. Since $q' \geq 2q+2$, $q'-q+1 \geq q+2$. Hence S'' does not contain any occurrences of $q+1$ or $q'-q-1$.

T' is constructed as the concatenation of sequences T and $-T$, where an element of T is in one of the ranges $[1, q]$ and $[q'-q, q'-1]$. Also, as before, since $q' \geq 2q+2$ we have $q'-q \geq q+2$. Hence T does not contain any occurrences of $q+1$ or $q'-q-1$. Now consider $-T$. It follows immediately that the elements of $-T$ are in the same ranges as T. Hence T' will not contain any instances of $q+1$ or $q'-q-1$.

It remains to show that $w_{q'}(U') = 0$. From Corollary 3.9 we know that $w_{q'}(U) = 0$. The only changes made to U are to add $q+1$ to half of the zeros and $q'-q-1$ to the other half. Thus, if U contains $2s$ zeros, $w(U') \equiv w(U) \equiv 0 + s(q+1+q'-q-1) \equiv 0 \pmod{q'}$, and the result follows. □

Remark 4.1. A good $\mathcal{SOS}_{q'}(n)$ with identical parameters could be constructed by taking an $\mathcal{SOS}_{q'-1}(n)$ constructed according to Corollary 3.9, and 'adding one' to every element. More formally, since each element of U is in $\mathbb{Z}_{q'-1}$, we treat every element of U as an integer, add one, and then treat the result as an element of $\mathbb{Z}_{q'}$.

The following example is similar to Examples 3.1 and 3.4.

Example 4.1. Suppose $q = 5$, $q' = 12$ and $n = 2$. As previously, we build upon the $\mathcal{OS}_5(2)$ with ring sequence $S = [01234\ 02413]$. Analogously to the second part of Example 3.1 we have

$$S'' = [0, 1, 2, 3, 4,\ \ 0, 2, 4, 1, 3,\ \ 0, 11, 10, 9, 8,\ \ 0, 10, 8, 11, 9].$$

Analogously to Example 3.3 we have

$$T' = [7, 1, 10, 3, 8,\ \ 5, 10, 4, 11, 3,\ \ 5, 11, 2, 9, 4,\ \ 7, 2, 8, 1, 9].$$

We simply concatenate them to obtain

$$\begin{aligned}U =&[0,1,2,3,4,\ 0,2,4,1,3,\ 0,11,10,9,8,\ 0,10,8,11,9,\\ &7,1,10,3,8,\ 5,10,4,11,3,\ 5,11,2,9,4,\ 7,2,8,1,9]\end{aligned}$$

which by Corollary 3.9 is an $\mathcal{SOS}_{12}(2)$ (and we can observe it contains no occurrences of $q+1=6$).

Finally, we replace every 0 with $q+1=6$ (since in this case $q'-q-1=q+1$) to obtain

$$\begin{aligned}U' =&[6,1,2,3,4,\ 6,2,4,1,3,\ 6,11,10,9,8,\ 6,10,8,11,9,\\ &7,1,10,3,8,\ 5,10,4,11,3,\ 5,11,2,9,4,\ 7,2,8,1,9]\end{aligned}$$

which by Theorem 4.2 is a good $\mathcal{SOS}_{12}(2)$ with $w_{12}(U')=0$.

Corollary 4.3. *There exists a good $\mathcal{SOS}_q(2)$ of period:*

$$\begin{aligned}&\frac{(q-2)(q-4)}{2} \quad \text{if } q\equiv 0 \pmod 4,\\ &\frac{(q-3)(q-5)}{2} \quad \text{if } q\equiv 1 \pmod 4,\\ &\frac{(q-2)(q-6)}{2} \quad \text{if } q\equiv 2 \pmod 4,\\ &\frac{(q-3)(q-7)}{2} \quad \text{if } q\equiv 3 \pmod 4,\end{aligned}$$

for all $q \geq 6$.

Proof. Suppose $q \geq 6$.

- If $q\equiv 0 \pmod 4$ then $\frac{q-2}{2}$ is odd, and by [9, Lemma 2.2] there exists an $\mathcal{OS}_{\frac{q-2}{2}}(2)$ with period $\frac{q-2}{2}(\frac{q-2}{2}-1)/2$. So by Theorem 4.2 there exists a good $\mathcal{SOS}_q(2)$ of period $4\frac{q-2}{2}(\frac{q-2}{2}-1)/2=\frac{(q-2)(q-4)}{2}$.
- If $q\equiv 1 \pmod 4$ then $\frac{q-3}{2}$ is odd, and by [9, Lemma 2.2] there exists an $\mathcal{OS}_{\frac{q-3}{2}}(2)$ with period $\frac{q-3}{2}(\frac{q-3}{2}-1)/2$. So by Theorem 4.2 there exists a good $\mathcal{SOS}_q(2)$ of period $4\frac{q-3}{2}(\frac{q-3}{2}-1)/2=\frac{(q-3)(q-5)}{2}$.
- If $q\equiv 2 \pmod 4$ then $\frac{q-2}{2}$ is even, and by [9, Lemma 2.2] there exists an $\mathcal{OS}_{\frac{q-2}{2}}(2)$ with period $\frac{q-2}{2}(\frac{q-2}{2}-2)/2$. So by Theorem 4.2 there exists a good $\mathcal{SOS}_q(2)$ of period $4\frac{q-2}{2}(\frac{q-2}{2}-2)/2=\frac{(q-2)(q-6)}{2}$.
- If $q\equiv 3 \pmod 4$ then $\frac{q-3}{2}$ is even, and by [9, Lemma 2.2] there exists an $\mathcal{OS}_{\frac{q-3}{2}}(2)$ with period $\frac{q-3}{2}(\frac{q-3}{2}-2)/2$. So by Theorem 4.2 there exists a good $\mathcal{SOS}_q(2)$ of period $4\frac{q-3}{2}(\frac{q-3}{2}-2)/2=\frac{(q-3)(q-7)}{2}$.

□

4.3 Adjusting the weight

Just as was the case in the previous section, we need to modify the sequences we have just constructed to ensure the result has weight coprime to q'. We can employ an identical strategy to that described in Section 3.4.

Construction 4.2. Suppose $q > 4$ and $q' = 2q + 2$ or $q' = 2q + 3$. Set $x = 0$, $y = 1$ and $z = q - 1$ (which are distinct since $q > 2$). By inspection it also holds that $(x + q + 1) + y + z = 2q + 1$ is coprime to q' for all possible choices of q and q'.

Suppose S is an $\mathcal{OS}_q(2)$ of maximal period m (i.e. of period $q(q-1)/2$ (q odd) or $q(q-2)/2$ (q even)), such that its ring sequence has the form $[xyzx\ldots]$, which exists from Lemma 3.11 — also observing that since $q > 4$ the sequence will contain at least two occurrences of x. Construct U from S using the method of Corollary 3.9, and U' from U using Construction 4.1, ensuring that the first two zeros in U are changed to $q + 1$. Observe that, from the method of construction, the ring sequence for U' has the form $[q+1, 1, q-1, q+1, \ldots]$. Finally, construct U^{**} from U' by deleting the first three elements of its ring sequence.

Theorem 4.4. *Suppose $q > 4$ and $q' = 2q + 2$ or $q' = 2q + 3$. If U^{**} is constructed according to the method of Construction 4.2 then it is a good $\mathcal{SOS}_{q'}(2)$ of period $2q(q-1) - 3$ (q odd) or $2q(q-2) - 3$ (q even) where in every case $w_{q'}(U^{**})$ is coprime to q'.*

Proof. By Theorem 4.2, the sequence U' is an $\mathcal{SOS}_{q'}(2)$ of period $2q(q-1)$ (q odd) or $2q(q-1)$ (q odd) where $w_{q'}(U) = 0$. The result now follows immediately by observing that constructing U^{**} from U' does not add any new 2-tuples, that $w_{q'}(U^{**}) = q' - (2q + 1)$, and, $q' - (2q + 1)$ is coprime to q'. □

The following brief example shows the operation of this construction.

Example 4.2. Suppose $q = 5$, $q' = 12$ and $n = 2$. We need an $\mathcal{OS}_5(2)$ with ring sequence starting [0140...]. One possibility is [01402 13423]. As in the previous examples we have

$$S'' = S'||-S' = [0, 1, 4, 0, 2,\ 1, 3, 4, 2, 3,\ 0, 11, 8, 0, 10,\ 11, 9, 8, 10, 9].$$

Analogously to Example 3.3 we have

$$T' = [7, 1, 8, 5, 10,\ 1, 9, 4, 10, 3,\ 5, 11, 4, 7, 2,\ 11, 3, 8, 2, 9].$$

We simply concatenate them to obtain

$$\begin{aligned} U =& [0, 1, 4, 0, 2,\ 1, 3, 4, 2, 3,\ 0, 11, 8, 0, 10,\ 11, 9, 8, 10, 9, \\ & 7, 1, 8, 5, 10,\ 1, 9, 4, 10, 3,\ 5, 11, 4, 7, 2,\ 11, 3, 8, 2, 9] \end{aligned}$$

which by Corollary 3.9 is an $\mathcal{SOS}_{12}(2)$ (and we can observe it contains no occurrences of $q+1 = q' - q - 1 = 6$).

Next, we replace every 0 with $q+1 = 6$ (since in this case $q' - q - 1 = q+1$) to obtain

$$\begin{aligned}U' =& [6,1,4,6,2,\ 1,3,4,2,3,\ 6,11,8,6,10,\ 11,9,8,10,9,\\ & 7,1,8,5,10,\ 1,9,4,10,3,\ 5,11,4,7,2,\ 11,3,8,2,9]\end{aligned}$$

which by Theorem 4.2 is a good $\mathcal{SOS}_{12}(2)$ with $w_{12}(U') = 0$.

Finally we simply delete the first three terms of U' to obtain

$$\begin{aligned}U^{**} =& [6,2,\ 1,3,4,2,3,\ 6,11,8,6,10,\ 11,9,8,10,9,\\ & 7,1,8,5,10,\ 1,9,4,10,3,\ 5,11,4,7,2,\ 11,3,8,2,9].\end{aligned}$$

which is a good $\mathcal{SOS}_{12}(2)$ of period 37 with $w_{12}(U^{**}) = 1$.

The following result follows immediately from Corollary 4.3 and Theorem 4.4.

Corollary 4.5. *There exists a good $\mathcal{SOS}_q(2)$ of period:*

$$\begin{aligned}&\frac{(q-2)(q-4)}{2} - 3 \quad \text{if} \quad q \equiv 0 \pmod 4,\\ &\frac{(q-3)(q-5)}{2} - 3 \quad \text{if} \quad q \equiv 1 \pmod 4,\\ &\frac{(q-2)(q-6)}{2} - 3 \quad \text{if} \quad q \equiv 2 \pmod 4,\\ &\frac{(q-3)(q-7)}{2} - 3 \quad \text{if} \quad q \equiv 3 \pmod 4,\end{aligned}$$

for all $q \geq 12$, where in every case the weight of the sequence is a unit modulo q.

5 Constructing orientable sequences

We now consider how to obtain large-period orientable sequences using the special orientable sequences we have constructed earlier in this paper. We follow two different approaches, both employing recursive construction methods described in Alhakim et al. [2].

5.1 Special orientable sequences for $n = 3$

We first show how to generate an $\mathcal{SOS}_q(3)$ with large period for arbitrary $q > 2$. We do so using the following result[2]. In this case we do not require the input sequences to be good.

[2]Note that this is actually a special case of the result from [2].

Theorem 5.1 ([2], Theorem 6.11)**.** *Suppose $S = (s_i)$ is an $\mathcal{SOS}_q(n)$ of period m and $q > 2$. If $w_q(S)$ is coprime to q then the set $D^{-1}(S)$ contains cyclic shifts of a single $\mathcal{SOS}_q(n+1)$ of period qm (where D is as defined in Section 1.2).*

Combining this with Construction 3.5 and Theorem 3.12 we get the following corollary.

Corollary 5.2. *Suppose $q \geq 5$ and let S be an $\mathcal{OS}_q(2)$ of maximal period m (i.e. of period $q(q-1)/2$ (q odd) or $q(q-2)/2$ (q even)), such that its ring sequence has the form $[xyzx\ldots]$ or, should $x \neq 0$, the form $[0xyzx\ldots]$, which exists from Lemma 3.11, where x, y and z are as specified in Construction 3.5. Suppose U^* is constructed from S using the method of Construction 3.5, where $q' = 2q+1$ or $q' = 2q+2$. Then $D^{-1}(U^*)$ is a $\mathcal{SOS}_{q'}(3)$ of period $2q^3 - 2q^2 - 3q$ (q odd) or $2q^3 - 4q^2 - 3q$ (q even).*

Proof. By Theorem 3.12, U^* is a $\mathcal{SOS}_{q'}(2)$ of period $2q(q-1) - 3$ (q odd) or $2q(q-2) - 3$ (q even) where $w_{q'}$ is coprime to q'. The result follows from Theorem 5.1. □

We also have the following, which is immediate from Corollary 3.13.

Corollary 5.3. *There exists an $\mathcal{SOS}_q(3)$ of period:*

$$\begin{aligned} &\frac{q^3 - 6q^2 + 2q}{2} \quad \textit{if} \ \ q \equiv 0 \pmod 4, \\ &\frac{q^3 - 6q^2 - q}{2} \quad \textit{if} \ \ q \equiv 1 \pmod 4, \\ &\frac{q^3 - 8q^2 + 6q}{2} \quad \textit{if} \ \ q \equiv 2 \pmod 4, \\ &\frac{q^3 - 4q^2 - 3q}{2} \quad \textit{if} \ \ q \equiv 3 \pmod 4, \end{aligned}$$

for all $q \geq 11$.

Observe that these sequences have period a little less than the $\mathcal{OS}_q(3)$ sequences constructed in [9]. However, the sequences constructed here have the additional property of being both orientable and negative orientable, which may be of use in some applications.

5.2 Special orientable sequences for general n

We next show how to construct $\mathcal{SOS}_q(n)$ with large period for arbitrary $q > 3$ and arbitrary $n > 2$. We employ the following result[3]. Note that in

[3]As above, this is actually a special case of the result from [2].

this case we *do* require our input sequences to be good. We first need the following notation from [2]. Suppose that the ring sequence of a periodic sequence S is $[s_0, s_1, \cdots, s_{m-1}]$ and that r is the smallest non-negative integer such that $a^t = s_r, s_{r+1}, \cdots, s_{r+t-1}$ is a maximal run for $a \in \mathbb{Z}_q$, where a^t denotes a string of t consecutive terms a. Define the sequence $\mathcal{E}_a(S)$ to be the sequence with ring sequence

$$[s_0, s_1, \cdots, s_{r-1}, a, s_r, s_{r+1}, \cdots, s_{m-1}]$$

i.e. where the occurrence of a^t is replaced with a^{t+1}.

Theorem 5.4 ([2], Corollary 6.22)**.** *Suppose S_n is a good $\mathcal{SOS}_q(n)$ of period m_n, where $w_q(S_n)$ is coprime to q. Recursively define the sequences $S_{i+1} = \mathcal{E}_a(D^{-1}(S_i))$, where $a = 1 - w_q(D^{-1}(S_i))$, for $i \geq n$, and suppose S_i has period m_i ($i > n$). Then, S_i is an $\mathcal{SOS}_q(i)$ for every $i \geq n$, and $m_{n+j} = qm_{n+j-1} + 1$ for every $j \geq 1$ (and hence $m_{n+j} = q^j m_n + \frac{q^j-1}{q-1}$ for every $j \geq 1$).*

Combining this theorem with Construction 4.2 and Theorem 4.4 we get the following corollary.

Corollary 5.5. *Suppose $q \geq 5$ and let S be an $\mathcal{OS}_q(2)$ of maximal period m (i.e. of period $q(q-1)/2$ (q odd) or $q(q-2)/2$ (q even)), such that its ring sequence has the form $[xyzx\ldots]$ or, should $x \neq 0$, the form $[0xyzx\ldots]$, which exists from Lemma 3.11, where x, y and z are as specified in Construction 4.2. Suppose U^{**} is constructed from S using the method of Construction 4.2, where $q' = 2q+2$ or $q' = 2q+3$. Setting $S_2 = U^{**}$ in Theorem 5.4, S_n is a good $\mathcal{SOS}_{q'}(n)$ of period*

$$2q'^{n-2}(q(q-1)-3) + \frac{q'^{n-2}-1}{q'-1} \quad (q \text{ odd}), \text{ or}$$
$$2q'^{n-2}(q(q-2)-3) + \frac{q'^{n-2}-1}{q'-1} \quad (q \text{ even})$$

for every $i \geq 2$.

Proof. By Theorem 4.4, U^{**} is a good $\mathcal{SOS}_{q'}(2)$ of period $2q(q-1)-3$ (q odd) or $2q(q-2)-3$ (q even), where $w_{q'}$ is coprime to q'. The result follows from Theorem 5.4. □

Corollary 5.6. *There exists an $\mathcal{SOS}_q(n)$ of period:*

$$\begin{cases} \frac{q^n - 6q^{n-1} + 2q^{n-2}}{2} + \frac{q^{n-2}-1}{q-1} & \text{if } q \equiv 0 \pmod 4, \\ \frac{q^n - 8q^{n-1} + 9q^{n-2}}{2} + \frac{q^{n-2}-1}{q-1} & \text{if } q \equiv 1 \pmod 4, \\ \frac{q^n - 8q^{n-1} + 6q^{n-2}}{2} + \frac{q^{n-2}-1}{q-1} & \text{if } q \equiv 2 \pmod 4, \\ \frac{q^n - 10q^{n-1} + 15q^{n-2}}{2} + \frac{q^{n-2}-1}{q-1} & \text{if } q \equiv 3 \pmod 4, \end{cases}$$

for all $q \geq 12$, and $n \geq 2$.

Proof. Suppose $q \geq 12$ and $n \geq 2$.

- If $q \equiv 0 \pmod 4$ then $r = \frac{q-2}{2}$ is odd, and by Corollary 5.5 there exists a good $\mathcal{SOS}_q(n)$ with period $2q^{n-2}(r(r-1)-3)+\frac{q^{n-2}-1}{q-1}$. Substituting in $r = (q-2)/2$ the result follows.
- If $q \equiv 1 \pmod 4$ then $r = \frac{q-3}{2}$ is odd, and by Corollary 5.5 there exists a good $\mathcal{SOS}_q(n)$ with period $2q^{n-2}(r(r-1)-3)+\frac{q^{n-2}-1}{q-1}$. Substituting in $r = (q-3)/2$ the result follows.
- If $q \equiv 2 \pmod 4$ then $r = \frac{q-2}{2}$ is even, and by Corollary 5.5 there exists a good $\mathcal{SOS}_q(n)$ with period $2q^{n-2}(r(r-2)-3)+\frac{q^{n-2}-1}{q-1}$. Substituting in $r = (q-2)/2$ the result follows.
- If $q \equiv 3 \pmod 4$ then $r = \frac{q-3}{2}$ is even, and by Corollary 5.5 there exists a good $\mathcal{SOS}_q(n)$ with period $2q^{n-2}(r(r-2)-3)+\frac{q^{n-2}-1}{q-1}$. Substituting in $r = (q-3)/2$ the result follows.

□

6 Concluding remarks

In this paper we have constructed orientable sequences with the additional property that they are also negative orientable. We used an approach proposed in [2] to generate orientable sequences with large period of any order over an alphabet of any size using 'starter' sequences with this additional property. Whilst this yields sequences with shorter periods than general orientable sequences, the periods remain asymptotic to the optimal as the alphabet size increases and the additional property could be a benefit in some applications.

It remains an open problem to find constructions of orientable sequences with optimal periods.

References

[1] A. Alhakim and M. Akinwande. A recursive construction of nonbinary de Bruijn sequences. *Des. Codes Cryptogr.*, 60(2):155–169, 2011.

[2] A. Alhakim, C. J. Mitchell, J. Szmidt, and P. R. Wild. Orientable sequences over non-binary alphabets. *Cryptogr. Commun.*, 16:1309–1326, 2024.

[3] J. Burns and C. J. Mitchell. Coding schemes for two-dimensional position sensing. Technical Report HPL–92–19, January 1992. `https://www.chrismitchell.net/HPL-92-19.pdf`.

[4] Z.-D. Dai, K. M. Martin, M. J. B. Robshaw, and P. R. Wild. Orientable sequences. In M. J. Ganley, editor, *Cryptography and Coding III*, pages 97–115. Oxford University Press, Oxford, 1993.

[5] D. Gabrić and J. Sawada. Constructing k-ary orientable sequences with asymptotically optimal length, 2024. Available at `https://arxiv.org/abs/2407.07029`.

[6] D. Gabrić and J. Sawada. Construction of orientable sequences in $O(1)$-amortized time per bit, 2024. Available at `https://arxiv.org/abs/2401.14341`.

[7] A. Lempel. On a homomorphism of the de Bruijn graph and its application to the design of feedback shift registers. *IEEE Trans. Comput.*, **C-19**:1204–1209, 1970.

[8] C. J. Mitchell and P. R. Wild. Constructing orientable sequences. *IEEE Trans. Inform. Theory*, 68:4782–4789, 2022.

[9] C. J. Mitchell and P. R. Wild. Orientable and negative orientable sequences. *Discrete Appl. Math.*, 377:242–259, 2025.