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GROUP DIVISIBLE DESIGNS WITH DUAL PROPERTIES

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Abstract. Necessary and sufficient conditions are given for a square group divisible design to have a group divisible dual, and the structure of such designs is examined.

1. INTRODUCTION

The study of group divisible (or simply GD) designs commenced with the work of Bose and Connor [3] who introduced them as an important subclass of the partially balanced designs with two associate classes of Bose and Nair [4]. Much research has been done in connection with these designs since the appearance of [3] for a survey of known results see [6] or [10].

The purpose of this paper is to prove the following two results:

Theorem 1 If both \underline{D} and \underline{D}^* (the dual of \underline{D}) are GD, then the group divisions of \underline{D} and \underline{D}^* form a strong tactical division of \underline{D} , and either :

- (i) \underline{D} and \underline{D}^* are semi-regular group divisible, or
- (ii) $b=v$ and $\underline{D}, \underline{D}^*$ are regular group divisible.

Theorem 2 If \underline{D} is a square GD design, then the following are equivalent:

- (i) \underline{D} admits a strong tactical division whose point classes form the classes of the group division of \underline{D} .
- (ii) \underline{D}^* admits a λ -point division with the same number of classes as the group division of \underline{D} .
- (iii) \underline{D}^* is GD with the same parameters as \underline{D} .
- (iv) \underline{D}^* is GD.

Theorem 2 generalises a result due to Bose [2] who proved that (iii) implies (i). Other results have been obtained giving necessary and sufficient conditions for a square GD design \underline{D} to have a dual which is GD with the same parameters as \underline{D} . In particular note the

results of Connor [7] and Shrikhande and Bhagwandas [15]; for details see Result 5 below. However, not every square GD design has a GD dual; an example of a square semi-regular GD design with a non-GD dual can be found in [7], and an infinite family of square regular GD designs with non-GD duals can be found in the recent paper, [8], of Jungnickel and Vedder.

The results of this paper are contained in the author's thesis for the degree of Doctor of Philosophy at the University of London [9] written at Westfield College under the supervision of Professor F.C.Piper. Some of the work in this paper was done while the author was studying at the University of California at Los Angeles under th supervision of Professor B. Rothschild.

2. DEFINITIONS AND PRELIMINARY RESULTS

For definitions and results on t -designs see, for instance, [1] or [5]. Note that the definiton of t -design used in this paper excludes the possibility of repeated blocks or points, i.e. no two blocks (or points) are incident with the same set of points (or blocks).

It is straightforward to show that $bk = vr$ for any 1-design, and hence $b = v$ if and only if $r = k$. If, for a 1-design D , $b = v$, then D is said to be square. Also, if D is a 1-design, then D^* (the dual of D obtained from D by interchanging the roles of points and blocks) is also a 1-design. We will always assume that D is a 1-design with $1 < k < v - 1$ (and hence $1 < r < b - 1$).

An incidence matrix $A = (a_{ij})$ for D is a v by b matrix with its rows indexed by the points of D and its columns indexed by the blocks of D such that $a_{ij} = 1$ if the i^{th} point is incident with the j^{th} block and $a_{ij} = 0$ otherwise.

The connection number of two points of \underline{D} is the number of blocks incident with them both, and dually, the intersection number of two blocks is the number of points incident with them both. (If x, y are two blocks of \underline{D} , then we will often write $|x \cap y|$ for the intersection number of x and y .)

A point division of \underline{D} is a partition $\underline{P}_1, \dots, \underline{P}_d$ ($1 < d < v$) of the points of \underline{D} , such that the connection number of two distinct points from classes \underline{P}_i and \underline{P}_j depends only on the choice of i and j and is denoted by λ_{ij} . A λ -point division is a point division such that $\lambda_{ii} = \lambda$ for every i with $|\underline{P}_i| > 1$ (i.e. for every i with λ_{ii} defined). A group division of \underline{D} is a λ -point division having an associated constant λ' such that $\lambda_{ij} = \lambda'$ for every i, j ($i \neq j$).

We also assume that $\lambda \neq \lambda'$ for a group division or else \underline{D} is a 2-design (or a "B.I.B.D."). It can easily be shown that a group division must have a related constant l , ($1 < l < v$), such that $|\underline{P}_i| = l$ for every i . Since a group division of a 1-design (if it exists) is clearly unique, a 1-design admitting a group division is called group divisible (or just GD). Also, we will henceforth assume that $\lambda' > 0$ for a GD design (since, if $\lambda' = 0$, then \underline{D} is the disjoint union of d 2-(l, k, λ) designs).

If \underline{D} is GD and A is an incidence matrix for \underline{D} "associated with" the group division (i.e. the first l rows of A correspond to the points of \underline{P}_1 , the next l rows to \underline{P}_2 , etc.) then:

So, by inspection, if j is the all +1 row vector

$$e_{11} = (0 \ 0 \ \dots \ 0 \ 1 \ -1 \ 0 \ 0 \ \dots \ 0) \quad \text{and}$$

$$\frac{f_w}{w} = \begin{pmatrix} \leftarrow & (w-1)1 & \leftarrow & 1 & \rightarrow & \leftarrow & 1 & \rightarrow & \leftarrow & (d-w-1)1 & \leftarrow \\ (0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & -1 & -1 & \dots & -1 & 0 & 0 & \dots & 0) \end{pmatrix}; \text{ then:}$$

$$j(AA^T) = ((v-1)\lambda^2 + (1-v)\lambda + r) j$$

$$\underline{e}_u (\mathbf{A}\mathbf{A}^T) = (r-\lambda)\underline{e}_u \quad (1 \leq u \leq v-1; \quad u \neq tl \text{ for any } t)$$

$$\underline{f}_w(AA^T) = (r + (1-1)\lambda - 1\lambda') \underline{f}_w \quad 1 \leq w \leq d-1$$

But $jA = kj$ and $jA^T = rj$ and so $j(AA^T) = rkj$. Hence:

Result 1 (Bose and Connor, [3]) If D and A are as above then:

- (i) $v = d1$.

(ii) $(v-1)\lambda' + (1-1)\lambda = r(k-1)$.

iii) The eigenvalues of AA^T are: $(r-\lambda)$, $(rk-v\lambda')$ and rk with multiplicities $v-d$, $d-1$ and 1 respectively.

iv) $|AA^T| = rk(r-\lambda)^{v-d}(rk-v\lambda')^{d-1}$.

From Result 1, since AA^T is positive semi-definite, we have:

Result 2 (Bose and Connor, [3]) If \underline{D} is GD then $rk \geq v\lambda'$.

By definition of design, $r > \lambda$, and so if \underline{D} is GD, then $rk > v\lambda'$ if and only if AA^T is non-singular. This led Bose and Connor (in [3]) to classify GD designs as follows:

Result 3 If \underline{D} is GD then either:

(i) $rk = v\lambda'$ and $b \geq \text{rank } A = \text{rank } AA^T = v-d+1$ (in which case \underline{D} is said to be Semi-regular GD, or SRGD), or

$rk > v\lambda'$ and $b \geq \text{rank } A = \text{rank } AA^T = v$ (in this case \underline{D} is said to be Regular GD, or RGD).

In fact Bose and Connor's definition of GD designs allows two points to be incident with the same set of blocks, and hence allows the case $r = \lambda$. Such designs they call Singular GD (SGD) designs. The definition used here does not permit SGD designs, and in any it can be shown that an SGD design consists of a 2-design with each point repeated 1 times

We now consider the cases when the bounds of Result 3 are reached.

Result 4 (Roy and Laha [11, 12]) If \underline{D} is SRGD then $b = v-d+1$ if and only if \underline{D}^* is a 2-design. In this case \underline{D}^* is a $2-(b, r, k-r+\lambda)$ design.

Square RGD designs whose duals are not RGD with the same parameters as \underline{D} seem rare; in fact the only such designs known to the author are those given in [8], Theorem 1.6. In this connection we also have:

Result 5 (Connor [7], Shrikhande and Bhagwandas [15])

Let \underline{D} be a square GD design. If one of the conditions below is satisfied then \underline{D}^* is GD with the same parameters as \underline{D} .

(i) \underline{D} is RGD and $(k^2 - v\lambda', \lambda - \lambda') = 1$.

(ii) $|\lambda - \lambda'| = 1$.

In this paper we give some further necessary and sufficient conditions for a square GD design to have a GD dual.

If D^* admits a point division then we will denote the division by B_1, \dots, B_c and we will write the intersection numbers of D as ρ_{ij} . Furthermore, if D^* is GD then we will set $\rho = \rho_{ii}$ and $\rho' = \rho_{ij}$ ($i \neq j$).

A tactical decomposition of D is a partition of the points and blocks of D into classes P_1, \dots, P_d and B_1, \dots, B_c respectively ($1 < d < v$, $1 < c < b$), such that:

- (i) The number of points of P_i incident with a block of B_j depends only on i and j and is denoted by β_{ij} , and, dually
- (ii) The number of blocks of B_j incident with a point of P_i is a constant γ_{ij} depending only on the choice of classes.

A tactical division of D is a tactical decomposition whose point classes form a λ -point division of D . We then have:

Result 6 (Beker, [1]) Suppose $T(D)$ is a tactical division of D with c block classes and d point classes. Then:

$$b + d \geq v + c.$$

- (iii) The following are equivalent:

- (a) $b + d = v + c$ (and $T(D)$ is said to be strong).
- (b) The block classes of $T(D)$ form a point division of D^* .
- (c) Two distinct blocks from the same block class of $T(D)$ always have intersection number $k - r + \lambda$.

To prove our theorems we also require the following results on SRGD designs:

Result 7 (Bose and Connor, [3]) If D is GD then D is SRGD if and only if every block is incident with precisely k/d points of each point class.

Result 8 (Saraf, [13]) If D is SRGD and x, y are two distinct blocks of D , then $|x \cap y| = k - r + \lambda$ if and only if $|x \cap z| = |y \cap z|$ for every block z ($z \neq x$ or y).

Finally, we state a result on graphs.

An adjacency matrix $T = (t_{ij})$ of a graph G on v vertices is a v by v matrix with $t_{ij} = 1$ if the i^{th} and j^{th} vertices are adjacent and 0 otherwise. Then G is regular if and only if $jT = \theta_0 j$ where θ_0 is the valency of G .

Result 9 (see, for instance, [5]) Let T be an adjacency matrix of a regular graph G with valency θ_0 . If $T|j^{-1}$ has only two eigenvalues θ_1 and θ_2 ($\theta_1 > \theta_2$) then G is a $\Gamma(c,m)$ if and only if $\theta_0 = \theta_1$, where $\Gamma(c,m)$ is the disjoint union of c complete graphs on m vertices each. In this case $\theta_2 = -1$ and has multiplicity $v-m$.

3. THE MAIN RESULTS

To prove Theorems 1 and 2 we first require:

Lemma 1 Suppose P_1, \dots, P_d is a λ -point division of D and B_1, \dots, B_c is a ρ -point division of D^* with $\rho = k - r + \lambda$. Label the points of P_i : $P_{i1}, P_{i2}, \dots, P_{il_i}$ and the blocks of B_j : $x_{j1}, x_{j2}, \dots, x_{jm_j}$, ($|P_i| = l_i$ and $|B_j| = m_j$ throughout). Suppose also that there are $\beta_{i(jt)}$ points of P_i incident with block x_{jt} of B_j and $\gamma_{(is)j}$ blocks of B_j incident with point P_{is} of P_i . (Note that if P_1, \dots, P_d and B_1, \dots, B_c form a tactical decomposition of D , $\beta_{i(jt)} = \beta_{ij}$ and $\gamma_{(is)j} = \gamma_{ij}$ for every t and s). Then, for every choice of P_{is} and x_{jt} :

$$\sum_{u=1}^d \lambda u \beta_{u(jt)} = \sum_{w=1}^c \rho_{jw} \gamma_{(is)w}.$$

Proof Let A be an incidence matrix for D "associated with" the point divisions of D and D^* , and consider the matrix identity $A(A^T A) = (AA^T)A$.

The entry in row $\sum_{w=1}^{i-1} l_w + s$ and column $\sum_{w=1}^{j-1} m_w + t$ of $A(A^T A)$ equals

$\sum_{w=1}^c \rho_{wj} Y_{(is)w} + (k - \rho) \delta.$ The corresponding entry in $(AA^T)_A$ equals $\sum_{u=1}^d \lambda_{ui} \beta_{u(jt)} + (r - \lambda) \delta,$ where $\delta = \begin{cases} 1 & \text{if } P_{is} \text{ is incident with } x_{jt}, \\ 0 & \text{otherwise} \end{cases}$

Using the fact that $\rho = k - r + \lambda,$ the lemma follows. \square

We can now prove the two main theorems.

Theorem 1 If D and D^* are GD, then the group divisions of D and D^* form a strong tactical division of $D,$ and either:

- (i) D, D^* are SRGD; $\rho = k - r + \lambda,$ $\rho' = \lambda'v/b,$ $\beta_{ij} = k/d$ and $\gamma_{ij} = r/c$ for every $i, j;$ or
- (ii) $b = v;$ D, D^* are RGD with the same parameters and $\beta_{ij} = \gamma_{ij}$ for every $i, j.$

Proof (i) Suppose D is SRGD. Then, by Result 8, $\rho = k - r + \lambda.$

Using Result 7 and the notation of Lemma 1 we have:

$\lambda_{ui} = \lambda'$ ($u \neq i$), $\lambda_{ii} = \lambda,$ $\beta_{u(jt)} = k/d,$ $\rho_{wj} = \rho'$ ($w \neq j$) and $\rho_{jj} = \rho = k - r + \lambda.$ So, by Lemma 1:

$$k((d-1)\lambda' + \lambda)/d = \rho' \sum_{w=1}^c Y_{(is)w} + (\rho - \rho') Y_{(is)j}.$$

Clearly $\sum_{w=1}^c Y_{(is)w} = r$ and so $Y_{(is)j}$ is a constant independent of i and $s;$ hence $Y_{(is)j} = r/c$ for every i, s and $j.$ So, by Result 7, D^* is SRGD and the group divisions form a tactical division which is strong by Result 6. Finally $\rho' = \lambda'v/b$ by Result 3(i).

(ii) If D is RGD then D^* is RGD by (i) above. Hence, by Result 3(ii), $b = v$ and $r = k.$

Let A be an incidence matrix for D associated with the group divisions D and $D^*.$ For any real matrix X it is clear that XX^T and $X^T X$ have the same non zero eigenvalues with the same multiplicities.

Hence, from Result 1(iii), $A^T A$ has eigenvalues: k^2 , $k - \lambda$ and $k^2 - v\lambda'$ with multiplicities 1, $v - d$ and $d - 1$. Note also that k^2 clearly has as eigenvector j (or any multiple thereof).

We now define a graph G having as vertices the blocks of D and having two vertices adjacent if and only if the corresponding blocks have intersection number ρ . Now since D^* is GD , G is a $\Gamma(c, m)$, and $T = ((k - \rho')I + \rho'J - A^T A) / (\rho' - \rho)$ where T is an adjacency matrix for G .

Since $r, k > 1$ and $\lambda' \neq 0, k - \lambda$ and $k^2 - v\lambda'$ are distinct from k^2 , and so their corresponding eigenvectors for $A^T A$ are orthogonal to j . Hence T has eigenvalues: $((v-1)\rho' - k(k-1)) / (\rho' - \rho)$ the valency, $(\lambda - \rho') / (\rho' - \rho)$ and $((v\lambda' - \rho') - k(k-1)) / (\rho' - \rho)$ with multiplicities 1, $v-d$ and $d-1$ respectively.

Thus, since G is a $\Gamma(c, m)$, Result 9 gives either $\lambda - \rho' = (v - 1)\rho' - k(k - 1)$ or $v\lambda' - \rho' = (v - 1)\rho'$ and the group division of D^* has $v - d + 1$ or d classes respectively. But $1 < d < v$, d divides v and the number of classes divides v , and so the group division of D^* cannot have $v - d + 1$ classes, i.e. we must have $\lambda' = \rho'$, and, again by Result 9, $(\lambda - \rho') / (\rho' - \rho) = -1$ and so $\rho = \lambda$.

So if D is RGD then D^* is RGD with the same parameters as D . Finally we must show that the group divisions of D and D^* form tactical divisions of D . Using Lemma 1 we have (for every P_{is} and x_{jt}):

$$\sum_{u=1}^d \beta_{u(jt)} + (\lambda - \lambda') \beta_{i(jt)} = \lambda' \sum_{w=1}^c \gamma_{(is)w} + (\lambda - \lambda') \gamma_{(is)j}$$

$$\text{Hence, since } \sum_{u=1}^d \beta_{u(jt)} = \sum_{w=1}^c \gamma_{(is)w} = k,$$

$\beta_{i(jt)} = \gamma_{(is)j}$ for every choice of P_{is} and x_{jt} . So the group divisions form a tactical division which is strong by Result 6, and $\beta_{ij} = \gamma_{ij}$ for every i, j . \square

Theorem 2 If \underline{D} is a square GD design, then the following are equivalent :

- (i) \underline{D} admits a strong tactical division whose point classes form the classes of the group division of \underline{D} .
- (ii) \underline{D}^* admits a λ -point division with $c = d$.
- (iii) \underline{D}^* is GD with the same parameters as \underline{D} .
- (iv) \underline{D}^* is GD.

Proof (iii) \Rightarrow (iv) is trivial.

(iv) \Rightarrow (i) is immediate by Theorem 1.

(i) \Rightarrow (ii) follows by Result 6.

(ii) \Rightarrow (iii). By a similar argument to that used to obtain

Result 1 (ii) we have:

$$k(k-1) = \sum_{\substack{j=1 \\ j \neq i}}^d m_j \rho_{ij} + (m_i - 1)\lambda \text{ for every } i.$$

So, summing both sides over all blocks of \underline{D} we obtain:

$$\begin{aligned} \sum_{i=1}^d \sum_{\substack{j=1 \\ j \neq i}}^d m_i m_j \rho_{ij} &= vk(k-1) + v\lambda \sum_{i=1}^d m_i^2 \\ &= v\lambda + v^2 \lambda' - v\lambda' - \lambda \sum_{i=1}^d m_i^2 \quad (\text{using Result 1 (ii)}). \end{aligned}$$

Let A be an incidence matrix for \underline{D} associated with the group division of \underline{D} and the point division of \underline{D}^* , and consider the matrix identity $(A^T A)^2 = A^T (AA^T) A$. Computing the diagonal entries of $A^T A A^T A$ in two ways (as in Lemma 1 above) and then summing both sides over all blocks of \underline{D} we arrive at the identity:

$$\sum_{i=1}^d \sum_{\substack{j=1 \\ j \neq i}}^d m_i m_j \rho_{ij}^2 = v^2 \lambda'^2 - v\lambda'^2 + v\lambda^2 - \lambda^2 \sum_{i=1}^d m_i^2.$$

Combining the above two identities we obtain:

$$\sum_{i=1}^d \sum_{\substack{j=1 \\ j \neq i}}^d m_i m_j (\lambda' - \rho_{ij})^2 = (v\lambda - \sum_{i=1}^d m_i^2)(\lambda - \lambda')^2. \quad (1)$$

$$\text{L.H.S.} \geq 0 \text{ and so } v\lambda \geq \sum_{i=1}^d m_i^2$$

$$\begin{aligned} \text{But } 0 \leq \sum_{i=1}^d (m_i - 1)^2 &= \sum_{i=1}^d m_i^2 - 2v_1 + 1^2 d \\ &= \sum_{i=1}^d m_i^2 - v_1. \end{aligned}$$

$$\text{Hence } v_1 = \sum_{i=1}^d m_i^2.$$

Then, substituting in (1) above, we obtain $\lambda' = p_{ij}$ for every i, j ($i \neq j$) and the result follows. \square

Remark The chief significance of this theorem is that (ii) implies (iv) implies (iii). It generalises the work of Bose [2] who showed that (iii) implies (i).

4. CONCLUDING REMARKS

It is clear from the results above that the restriction that the dual of a GD design be GD places powerful constraints on the structure of such a design. As we noted above, square RGD designs whose duals are not also GD seem rare. However, apart from Result 5 above (and a result implicit in [14] stating that a square GD design having $\lambda' = 1$ must have a GD dual), no results are known giving parameter sets for which such a design cannot exist.

Much work clearly remains to be done on this topic

REFERENCES

- H.J. Beker, "Strongly divisible 1-designs", Geom. Dedicata 8 (1979), 299-316.
- R.C. Bose, "Symmetric group divisible designs with the dual property", J. Stat. Plann. Inference 1 (1977), 87-101.
- [3] R.C. Bose and W.S. Connor, "Combinatorial properties of group divisible incomplete block designs", Ann. Math. Statist. 23 (1952), 367-383.
- [4] R.C. Bose and K.R. Nair, "Partially balanced incomplete block designs", Sankhya 4 (1939), 337-372.
- [5] P.J. Cameron and J.H. van Lint, "Graph theory, coding theory and block designs", L.M.S. Lecture Note Series 19, Cambridge University Press (1975).
- [6] W.H. Clatworthy, "Tables of two-associated-class partially balanced designs", Nat. Bur. Standards Appl. Math. Ser. No. 63 (1973).
- [7] W.S. Connor, "Some relations among the blocks of symmetrical group divisible designs", Ann. Math. Statist. 23 (1952), 602-609.
- [8] D. Jungnickel and K. Vedder, "Square divisible designs with $k = (n+1)\mu$ ", Arch. Math. (to appear)
- [9] C.J. Mitchell, "On divisions and decompositions of 1-designs", Ph.D. Thesis, University of London (1979).
- D. Raghavarao, "Constructions and combinatorial problems in design of experiments", John Wiley and Sons, New York (1971).
- J. Roy and R.G. Laha, "Classification and analysis of linked block designs", Sankhya 17 (1956), 115-132.

- [12] J. Roy and R.G. Laha, "On partially balanced linked block designs", Ann. Math. Statist. 28 (1957), 488-493.
- [13] W.S. Saraf, "On the structure and combinatorial properties of certain semi-regular group divisible designs", Sankhya (A) 23 (1961), 287-296.
- [14] S.S. Sane, "The existence problem for a family of divisible partial designs", J. Combin. Theory Ser. A. (to appear).
- [15] S.S. Shrikhande and Bhagwandas, "Duals of incomplete block designs", J. Indian Statist. Assoc. 3 (1965), 30-37.

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