

# SWITCH MINIMISATION FOR A SIMPLE BIDIRECTIONAL NETWORK

*Indexing terms: Switches, Switching and switching circuits, Networks*

The letter describes a method for generating a one-stage, one-sided switching network with the minimum number of switches for any required level of connectivity. The fact that the method generates optimal networks is established by proving appropriate lower bounds. This generalises recent work of Newbury.

**Introduction:** A number of recent papers<sup>1-4</sup> have considered the problem of minimising the number of switches for a certain type of one-stage, one-sided, rearrangeable switching network. In these papers it was shown that, if such a network is used to connect  $2n$  subscribers, then at least  $n^2 + 2n - 1$  switches are required. Moreover, suitable networks can be devised having this number of switches for every value of  $n$ . For small bidirectional networks this appears to be a potentially useful technique.

In a recent letter,<sup>5</sup> Newbury considered the minimum number of switches for the same type of network with one modification; he relaxes the constraint that  $n$  connections must be possible simultaneously. When  $n = 4$  he then gives the minimum number of switches for each possible case, where these bounds have been deduced from exhaustive computer search. In this letter we give mathematical proofs of Newbury's bounds, extend them to cover all values of  $n$ , and construct switching networks having this minimum number of switches for every possible case.

**Notation and definitions:** We first review the relevant definitions. We are concerned with the situation where  $2n$  subscribers are connected using  $k$  crosswires ( $k \leq n$ ), where each subscriber is connected by a switch to some (or all) of the crosswires. For convenience let the  $2n$  subscribers be labelled:

$$P_1, P_2, \dots, P_{2n}$$

and let the crosswires be labelled:

We call such an arrangement a 1-stage, 1-sided switching network.

If the network has the following additional property then we call it rearrangeably nonblocking (or just 'rearrangeable'). For every set of  $k$  disjoint pairs of subscribers

$$\{P_{f(1)}, P_{f(2)}\}, \{P_{f(3)}, P_{f(4)}\}, \dots, \{P_{f(2k-1)}, P_{f(2k)}\}$$

say (where  $f$  is a one-to-one function from  $\{1, 2, \dots, 2k\}$  into  $\{1, 2, \dots, 2n\}$ ), there exists an ordering of the  $k$  crosswires

$$X_{g(1)}, X_{g(2)}, \dots, X_{g(k)}$$

say (where  $g$  is a permutation of  $1, 2, \dots, k$ ), such that both the subscriber crosswire pairs

$$(P_{f(2i-1)}, X_{g(i)}) \text{ and } (P_{f(2i)}, X_{g(i)})$$

are connected by a switch for every  $i$  ( $1 \leq i \leq k$ ). That is, for every possible set of  $k$  telephone calls, each call can be assigned to a unique crosswire having switches in the appropriate two places. However, if some calls cease and the corresponding subscribers need to be reconnected in a different way, some rearrangement of existing calls onto different crosswires may be necessary (hence the term 'rearrangeably' nonblocking).

If a network satisfies the above property then we call it a 1-stage, 1-sided rearrangeable switching network for  $2n$  subscribers and  $k$  crosswires, or an  $\text{RSN}(n, k)$  for short. For every positive integer  $n$ , and every  $k$  ( $1 \leq k \leq n$ ), we are concerned with finding the minimum number of switches for an  $\text{RSN}(n, k)$ . Examples of an  $\text{RSN}(4, 1)$ , and  $\text{RSN}(4, 2)$  and an  $\text{RSN}(4, 3)$ ,

each with the minimum number of switches, are given in Newbury's recent letter.<sup>5</sup>

Note that, for the case  $n = k$ , the above definitions correspond precisely to those used earlier by Mitchell and Wild.<sup>1,2</sup> An  $\text{RSN}(n, n)$  is therefore the same as an  $\text{R2BSN}(n)$ ,<sup>1</sup> or an  $\text{RSN}(n)$ .<sup>2</sup>

**Theoretical results:** We first give two lemmas generalising lemmas 1 and 2 of Reference 1.

**Lemma 1:** In an  $\text{RSN}(n, k)$  every crosswire must be connected by switches to at least  $2n + 1 - k$  subscribers.

**Proof:** Suppose crosswire  $x$  is connected by switches to fewer than  $2n + 1 - k$  subscribers, i.e. there are at least  $k$  subscribers not connected to  $x$  by a switch. Let

$$Q_1, Q_2, \dots, Q_k$$

be  $k$  such subscribers. Let

$$R_1, R_2, \dots, R_k$$

be  $k$  subscribers distinct from  $Q_1, \dots, Q_k$ , and consider the following  $k$  disjoint pairs of subscribers:

$$\{Q_1, R_1\}, \{Q_2, R_2\}, \dots, \{Q_k, R_k\}$$

For each pair, crosswire  $x$  is not connected by a switch to at least one of the two subscribers, and hence we have a contradiction. The lemma follows.

**Lemma 2:** In an  $\text{RSN}(n, k)$ , at most one crosswire is connected by switches to precisely  $2n + 1 - k$  subscribers.

**Proof:** Suppose crosswires  $x$  and  $y$  are both connected to exactly  $2n + 1 - k$  subscribers. Let  $T$  be the set of subscribers connected to  $x$  by a switch and let  $U$  be the set of subscribers connected to  $y$  by a switch. Suppose there are  $s$  subscribers in both  $T$  and  $U$ ; then there are  $2n + 1 - k - s$  subscribers in  $T$  and not in  $U$  (and vice versa) and  $2k + s - 2n - 2$  subscribers in neither  $T$  nor  $U$ .

Let

$$Q_1, Q_2, \dots, Q_{2n+1-k}$$

be the subscribers in  $T$  which are not in  $U$ , and let

$$R_1, R_2, \dots, R_{2n+1-k-s}$$

be the subscribers in  $U$  which are not in  $T$ . Next let

$$W_1, W_2, \dots, W_{2k+s-2n-2}$$

be the subscribers which are in neither  $T$  nor  $U$ , and let

$$X_1, X_2, \dots, X_{2k+s-2n}$$

be any collection of  $2k + s - 2n$  subscribers which are both  $T$  and  $U$ . Now consider the following  $k$  disjoint pairs of subscribers:

$$\begin{aligned} &\{Q_1, R_1\}, \{Q_2, R_2\}, \dots, \{Q_{2n+1-k-s}, R_{2n+1-k-s}\} \\ &\{W_1, X_1\}, \{W_2, X_2\}, \dots, \{W_{2k+s-2n-2}, X_{2k+s-2n-2}\} \\ &\{X_{2k+s-2n-1}, X_{2k+s-2n}\} \end{aligned}$$

It is straightforward to see that, apart from the last pair, no pair of subscribers is contained in either  $T$  or  $U$ . This immediately gives a contradiction and the lemma follows.

Using these two lemmas we have the following theorem.

**Theorem 3:** In any  $\text{RSN}(n, k)$  there are always at least

$$2nk - (k - 1)^2$$

switches.

If we put  $k = n$ , we obtain theorem 3 of Reference 1 as an immediate corollary. In addition, if we put  $n = 4$ , we obtain the bound given by Newbury.<sup>5</sup> We now show how an RSN( $n, k$ ) can be constructed which has this minimum number of switches for every  $n$  and  $k$  ( $1 \leq k \leq n$ ).

*Construction method:* We now show how to construct an 'optimal' RSN( $n, k$ ). First, construct an RSN( $k, k$ ) having the minimum number of switches, i.e.  $k^2 + 2k - 1$ , using existing methods.<sup>1</sup> To this network add a further  $2n - 2k$  subscribers, and connect every one of these new subscribers to the existing  $k$  crosswires by a switch. The fact that this results in an RSN( $n, k$ ) is straightforward to establish. Moreover, this network has a total of

$$(k^2 + 2k - 1) + (2n - 2k)k = 2nk - k^2 + 2k - 1$$

switches. This is the minimum established in theorem 3.

*Assignment procedure:* It has been established<sup>2</sup> that the 'augmenting matching algorithm' can be used to rearrange connections in an RSN( $n, n$ ) so that new connections may be made without interrupting existing connections. The same method will work in an RSN( $n, k$ ) with  $k < n$ .

Also described in Reference 2 is an assignment procedure for the minimal RSN( $n, n$ ) constructed in that Reference, which makes correct choice of connection route without backtracking, provided all connections are known in advance. This procedure may be adapted to give a procedure with the same property for the minimal RSN( $n, k$ ) constructed from a minimal RSN( $k, k$ ) as above. This may be done by initially applying the first part of the original procedure to those pairs of subscribers contained in the embedded RSN( $k, k$ ). As the subscribers of the RSN( $n, k$ ) which are not in the embedded RSN( $k, k$ ) are connected by a switch to all  $k$  crosswires, there is no difficulty in assigning crosswires to the remaining subscriber pairs (as in the second part of the original procedure),

and there will be no need for backtracking.

*Concluding remarks:* We have therefore completely solved the problem of constructing switch-minimal networks of the type described. However, this does not completely solve the switch minimisation problem, as has been observed elsewhere.<sup>5,6</sup> It is conceivable, although perhaps unlikely, that the same degree of connectivity could be achieved with fewer switches if more than  $k$  crosswires are used: this would remove the requirement for every crosswire to be used when connecting  $k$  pairs of subscribers. For certain small cases this has been shown not to work,<sup>5,6</sup> although this is by no means conclusive. It is certainly an interesting and nontrivial topic for further research.

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