SWITCHING NETWORKS FOR BIDIRECTIONAL TELEPHONE SYSTEMS
Chris Mitchell and Peter Wild

Abstract
We consider two-stage bidirectional switching networks which have a minimum number of switches. Results on the structure of such networks in terms of the number of switches per crosswire are established.

1. Introduction

Newbury and Raby [2] have considered two-stage switching arrangements in bidirectional telephone systems. In this kind of network each subscriber line is connected via switches to some (or all) of a set of crosswires. Thus a connection between two subscribers is made along one crosswire via two switches. We call such an arrangement of subscriber lines, crosswires and switches connecting them, a two-stage bidirectional switching network.

Such a network may be represented as an incidence structure $(P,B,S)$ where $P = \{ P_1, \ldots, P_m \}$ is a set labelling the subscriber lines, $B = \{ x_1, \ldots, x_b \}$ is a set labelling the crosswires and $S \subseteq P \times B$ with $(P_i, x_j) \in S$ if and only if subscriber line $P_i$ is connected to crosswire $x_j$ by a switch. We only consider networks in which each crosswire has at least two switches on it.

Since pairs of subscriber lines must be connected via distinct crosswires and we would like to be able to connect as many subscribers in pairs as possible, we also require that $b > \lceil m/2 \rceil$. If $m = 2n + \delta$, where $\delta$ is 0 or 1, and $b = n + t$, where $t > 0$, we use $2BSN(n,t,\delta)$ to denote such a two-stage bidirectional switching network. The two-stage bidirectional switching networks with $2n$ subscriber lines
and \( n \) crosswires denoted by \( 2BSN(n) \) in [1] are here denoted by \( 2BSN(n,0,0) \).

Of interest is the minimum number of switches required to ensure that for any \( n \) disjoint pairs of subscriber lines there is an assignment of distinct crosswires to the pairs such that each pair may be connected via the crosswire assigned to it. A \( 2BSN(n,t,5) \) with this property is called rearrangeably non-blocking (or rearrangeable) and is denoted \( R2BSN(n,t,5) \). In this paper we consider \( R2BSN(n,t,5) \) with \( t = 0 \) and \( t = 1 \), and show that the minimum number of switches requirement puts strong conditions on the number of switches that the individual crosswires may contain.

2. \( R2BSN(n,0,5) \)

We extend the results of [1] to cover the case \( 5 \)

**Lemma 2.1.** Let \( D = (P,B,S) \) be an \( R2BSN(n,0,5) \). Then for each crosswire in \( B \) there are at most \( n-1 \) subscriber lines not connected to it by a switch.

**Proof.** Suppose \( x_j \in B \) is not connected to \( n \) subscriber lines. Without loss of generality we may assume these are \( P_1, \ldots, P_n \). Then \((P_1,P_{n+1}), \ldots, (P_n,P_{2n})\) is a collection of \( n \) disjoint pairs of subscriber lines such that crosswire \( x_j \) does not connect any of them. Since \( B \) contains only \( n \) crosswires these \( n \) disjoint pairs cannot be connected via distinct crosswires. But \( D \) is rearrangeable, so no such \( x_j \) exists. \( \square \)

**Lemma 2.2.** Let \( D = (P,B,S) \) be an \( R2BSN(n,0,5) \). Then there is at most one crosswire in \( B \) which is not connected to \( n-1 \) subscriber lines by a switch.

**Proof.** Suppose there are two such crosswires, \( x_1, x_2 \) say. Without loss of generality we may assume that \( x_1 \) is not connected to \( P_1, \ldots, P_{n-1} \) and \( x_2 \) is not connected to \( P_1, \ldots, P_s, P_n, \ldots, P_{2n-2-s} \), where \( 0 < s < n-1 \). Then
(P₁,P₂n−1−s), ..., (Pₛ,P₂n−2) and (Pₛ+1,Pₙ), ..., (Pₙ−1,P₂n−2−s) form a collection of n−1 disjoint pairs of subscriber lines such that neither x₁ nor x₂ connects any of them. Since B contains only n crosswires these disjoint pairs cannot be connected via distinct crosswires. But D is rearrangeable, and so there can exist at most one such crosswire.

PROPOSITION 2.3. Let D = (P,B,S) be an R2BSN(n,0,δ). Then |S| ≥ n² + (2+δ)n − 1. If equality holds then exactly one crosswire contains n + 1 + δ switches, and the other n−1 crosswires each contain n + 2 + δ switches.

Proof. By Lemma 2.1 each crosswire contains at least 2n + δ − (n−1) = n + 1 + δ switches, and by Lemma 2.2 at most one has exactly n + 1 + δ switches. Thus |S| ≥ (n+1+δ) + (n−1)(n+2+δ) = n² + (2+δ)n − 1. □

In [1] an R2BSN(n,0,0) with n² + 2n − 1 switches is constructed for every n. The following result shows that there also exist R2BSN(n,0,1) with n² + 3n − 1 switches for every n.

PROPOSITION 2.4. Let D = (P,B,S) be an R2BSN(n,0,0). Let P' = P ∪ {P₂n+1}, B' = B, and S' = S ∪ {(P₂n+1,xj) : xj ∈ B}. Then D' = (P',B',S') is an R2BSN(n,0,1) with |S'| = n²+3n−1.

Proof. Let A₁, ..., Aₙ be n disjoint pairs of subscriber lines in P'. First suppose P₂n+1 ∉ A₁ ∪ ... ∪ Aₙ. Then A₁, ..., Aₙ are n disjoint pairs of subscriber lines in P and so there are n distinct crosswires in B = B' which connect them.

Now suppose P₂n+1 ∈ A₁ ∪ ... ∪ Aₙ. Without loss of generality we may assume that A₁ = (P₁,P₂n+1) and that P' = (A₁ ∪ ... ∪ Aₙ) = (P₂n). Put A₁' = (P₁,P₂n). Then A₁', A₂, ..., Aₙ are n disjoint pairs of subscriber lines in P. Hence they are connected via distinct crosswires. Since the crosswire connecting P₁ and P₂n also connects P₁ and
P_{2n+1}, the n disjoint pairs A_1, \ldots, A_n are also connected via distinct crosswires. It follows that D' is rearrangeable and so is an R2BSN(n,0,1).

3. R2BSN(n,1,\delta)

We consider the structure of an R2BSN(n,1,\delta) having at most \(n^2 + (2+\delta)n - 1\) switches, i.e. the minimum number of switches for an R2BSN(n,0,\delta).

**Lemma 3.1.** Let \(D = (P,B,S)\) be an R2BSN(n,1,\delta). Then there is at most one crosswire in \(B\) which is not connected to \(n\) subscriber lines by a switch.

**Proof.** Suppose there are two such crosswires. Without loss of generality we may suppose that \(x_1\) is not connected to \(P_1, \ldots, P_n\) and \(x_2\) is not connected to \(P_{s+1}, \ldots, P_{2n-s}\) where \(0 \leq s \leq n\). Then \(\{P_1/P_{2n+1-s}\}, \ldots, \{P_s/P_{2n}\}\) and \(\{P_{s+1}/P_{n+1}\}, \ldots, \{P_n/P_{2n-s}\}\) are \(n\) disjoint pairs of subscriber lines none of which are connected via \(x_1\) or \(x_2\). Since \(B\) contains only \(n+1\) crosswires, these \(n\) disjoint pairs cannot be connected via distinct crosswires. But \(D\) is rearrangeable, so there is at most one such crosswire. \(\square\)

An R2BSN(n,1,\delta) in which every crosswire contains at least \(n+1+\delta\) switches has at least \((n+1)(n+1+\delta) = n^2 + (2+\delta)n + 1+\delta\) switches. Thus, in an R2BSN(n,1,\delta) with at most \(n^2 + (2+\delta)n - 1\) switches, some crosswire, \(x_1\) say, is connected to \(s \leq n-1\) subscriber lines. Moreover there must be at least \(s+1\) other crosswires which contain exactly \(n+1+\delta\) switches.

**Lemma 3.2.** Let \(D = (P,B,S)\) be an R2BSN(n,1,\delta) with at most \(n^2 + (2+\delta)n - 1\) switches. Suppose crosswire \(x_1\) contains \(s \leq n-1\) switches, and two crosswires, each containing \(n+1+\delta\) switches are connected to \(r\) common subscriber lines. Then \(3+\delta < r < n+\delta\) and \(s > \min(n+1+\delta, n+2+\delta-r)\).
Proof. Put $u = n + 1 + 5$. We note that at least $s + 1 > 3$ crosswires contain $u$ switches. Without loss of generality suppose that $x_2$ is connected to $P_1, \ldots, P_u$ and $x_3$ is connected to $P_1, \ldots, P_r, P_{u+1}, \ldots, P_{2u-r}$. We first show that $3 + 5 < r < n + 5$.

As $2u-r = 2n + 2 + 25 - r < 2n + 5$, we have $r > 2 + 5$. If $r = 2 + 5$ then for any permutation $f$ of 

\{ $u+1, \ldots, 2n+5$ \} the n pairs $(P_1,P_2), (P_3+5,P_f(u+1)), \ldots (P_u,P_f(2n+5))$ are disjoint, and only the pair $(P_1,P_2)$ is connected via $x_2$ or $x_3$. Since $D$ is rearrangeable the crosswire $x_1$ must connect one of the pairs $(P_1,P_f(n-1+i))$. Since this is true for every $f$ it follows that $x_1$ is connected to at least $n$ of the subscriber lines $P_{3+5}, \ldots, P_{2n+5}$. But $x_1$ contains only $s < n-1$ switches. Hence $r > 3 + 5$.

Now $r < u = n + 1 + 5$. If $r = n + 1 + 5$ then for any $(2+5)-subset A$ of $\{1,\ldots,u\}$ and any bijective mapping $f : \{1,\ldots,u\}-A \rightarrow \{u+1,\ldots,2n+5\}$ the pairs $(P_i,P_j), i,j \in A$, and $(P_s,P_f(s)), s \in \{1,\ldots,u\}-A$, are disjoint, and only the pair $(P_1,P_2)$ is connected via $x_2$ or $x_3$. It follows that $x_1$ connects one of the pairs $(P_s,P_f(s))$ for every $A$ and $f$, and hence contains at least $n + 2 + 5$ switches. But $x_1$ contains only $s < n-1$ switches and hence $r < n + 5$.

Now suppose $3 + 5 < r < n + 5$. Then for any $(2+5)-subset A$ of $\{1,\ldots,r\}$ and bijective mappings $f : \{1,\ldots,r\}-A \rightarrow \{2u-r+1,\ldots,2n+5\}$ and $g : \{r+1,\ldots,u\} \rightarrow \{u+1,\ldots,2u-r\}$, the n pairs $(P_i,P_j), i,j \in A$, $(P_s,P_f(s)), s \in \{1,\ldots,r\}-A$, and $(P_s,P_g(s)), s \in \{r+1,\ldots,u\}$, are disjoint, and only the pair $(P_1,P_2)$ are connected via $x_2$ or $x_3$. Hence for every $A$, $f$ and $g$, the crosswire $x_1$ connects one of the pairs $(P_s,P_f(s))$ or one of the pairs $(P_s,P_g(s))$. It follows that $x_1$ is either connected to at least $r+1+5$ of the subscriber lines $P_1, \ldots, P_r, P_{2u-r+1}, \ldots, P_{2n+5}$, or to at least
Thus \( u-r+1 = n+2+\delta-r \) of \( P_{r+1}, \ldots, P_{2u-r} \). Thus \( s > \min(r+1+\delta, n+2+\delta-r) \).

Lemma 3.2 says that in an \( R2BSN(n,1,\delta) \) with at most \( n^2 + (2+\delta)n - 1 \) switches, in which \( x_1 \) contains \( s < n-1 \) switches, any two of the \( s+1 \) or more crosswires which contain \( n+1+\delta \) switches are connected either to at most \( s-1 \) common subscriber lines or to at least \( n+2+\delta-s \) common subscriber lines. An exhaustive search has shown that no such \( R2BSN(n,1,\delta) \) exists for \( n \leq 6 \). The following questions remain to be answered:

1. Does there exist an \( R2BSN(n,1,\delta) \) with fewer switches than the minimal \( R2BSN(n,0,\delta) \) (which contains \( n^2 + (2+\delta)n - 1 \) switches)?

2. Does an \( R2BSN(n,t,\delta) \) with the minimum number of switches have \( t = 0 \)?

References


Chris Mitchell
Hewlett-Packard Ltd.
Filton Road
Stoke Gifford
Bristol BS12 6QZ
England

Peter Wild
Royal Holloway and Bedford New College
Egham Hill
Egham
Surrey TW20 0EX
England