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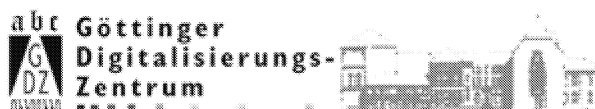
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Tactical decompositions of designs

HENRY BEKER, CHRISTOPHER MITCHELL AND FRED PIPER

0. Introduction

In this paper we have attempted to give an account of the theory of special decompositions of designs. We assume that the reader has some familiarity with incidence structures and their representation by incidence matrices.

The origin of this work could be considered to be the 1942 paper of Bose [11], in which the theory of resolvable and affine designs is introduced. These concepts were generalised by Shrikhande and Raghavarao in [41] to α -resolvable and affine α -resolvable designs. These results, together with the work of Dembowski on tactical decompositions (see [19], [20]), led naturally to the work of Beker and Mitchell on divisible and strongly divisible 1-designs.

Also of fundamental importance to the study of strongly divisible 1-designs is the significance of the intersection number $k - n$, first recognised in the 1953 paper of Majumdar [30]. This develops into the concept of $(k - n)$ -decompositions and from there to the work of Singhi and Shrikhande [44] and Beker and Haemers [6]. As is shown below, every strongly divisible 2-design is a nontrivial example of a 2-design admitting a $(k - n)$ -decomposition, and these two classes of designs are thus closely related.

Tactical decompositions of designs are also of great interest because of their intimate connection with automorphism groups of designs. In particular, note that the point and block orbits of any automorphism group of a design give rise to a tactical decomposition of this design, and thus many results concerning tactical decompositions of designs have as immediate corollaries results about automorphism groups of designs.

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To conclude this introduction, we discuss briefly the contents of each of the eleven sections of this paper.

Sections 1, 2, 3 and 4 are introductory in nature. Section 1 contains a list of results on designs that are relevant to the discussions in later sections. In Section 2 we develop the idea of decompositions of incidence matrices, by means of which many of the later results can be obtained, and in Section 3 we give the corresponding definitions and results on decompositions of designs. To conclude this introductory portion of the paper, Section 4 gives the definition and some basic results on tactical decompositions of 1-designs.

In Sections 5, 6 and 7 we restrict ourselves to the case when the design admitting a special decomposition is a 2-design. Section 5 contains a discussion of strongly divisible 2-designs, while in Section 6 we specialise further to the special class of strongly divisible 2-designs, which we call here strongly resolvable 2-designs (in the notation of Shrikhande and Raghavarao these are strongly α -resolvable designs). In Section 7 we examine 2-designs admitting $(k - n)$ -resolutions, and, as mentioned above, we are at the same time also considering all designs having a strong tactical decomposition.

In Section 8 we consider strongly divisible 1-designs; many of the results on strongly divisible 2-designs generalise in a natural way to the 1-design case. Section 9 contains a discussion of orbit theorems for designs. Many of these results are immediate corollaries of earlier results on tactical decompositions of designs.

The penultimate section, Section 10, contains a list of construction methods for strongly divisible and strongly resolvable 2-designs. This list includes all the construction methods of such 2-designs known to the authors.

Finally, Section 11 contains some further, miscellaneous results concerning tactical decompositions of designs, including an outline of a way in which the theory of group divisible 1-designs has relevance to the theory of strongly divisible designs.

1. Preliminary results

A t -design D ($t \geq 0$) is an incidence structure with a finite, nonzero number of points and blocks (normally denoted by v and b , respectively) having the following properties:

(i) Every block of D is incident with precisely k points for some constant k ($0 < k < v$).

(ii) Every set of t distinct points is incident with exactly λ blocks for some constant $\lambda > 0$.

(iii) No two distinct blocks are incident with the same set of points.

(iv) No two distinct points are incident with the same set of blocks.

Any incidence structure satisfying (i), (iii) and (iv) will be called a *design*, and it

is clear that the terms 0-design and design are synonymous. If D contains v points and satisfies (i)–(iv) then we say that D is a $t - (v, k, \lambda)$ design.

By (iii) above we may, in a design, regard a block as the set of points incident with it, with incidence as set-theoretic inclusion. In any design we will call the number of points incident with a pair of blocks x, y the *intersection number* of x and y and, using the set-theoretic notation, we will often write $|x \cap y|$ for this number. Dually, if P, Q are two points of a design, then we will call the number of blocks incident with both P and Q the *connection number* of P and Q .

RESULT 1.1. *If D is a $t - (v, k, \lambda)$ design, then, for every s satisfying $0 \leq s \leq t$, D is an $s - (v, k, \lambda_s)$ design, where $\lambda_s = \binom{v-s}{t-s} / \binom{k-s}{t-s}$.*

In any t -design we have $b = \lambda_0$, and if $t \geq 1$ we set $r = \lambda_1$. As an immediate corollary of Result 1.1 we have:

RESULT 1.2. *Suppose D is a $t - (v, k, \lambda)$ design.*

- (i) *If $t \geq 1$, then $bk = vr$.*
- (ii) *If $t \geq 2$, then $\lambda_2(v - 1) = r(k - 1)$.*

A further basic result is:

RESULT 1.3 (Fisher's Inequality). *If D is a $2 - (v, k, \lambda)$ design then $b \geq v$.*

If $b = v$ for a $2 - (v, k, \lambda)$ design, then the design is *symmetric*. For this case we have:

RESULT 1.4. *Let D be a $2 - (v, k, \lambda)$ design. Then the following are equivalent:*

- (i) *D is symmetric.*
- (ii) *Every pair of blocks have intersection number λ .*
- (iii) *Every pair of blocks have constant intersection number.*

Because of the importance of the number $r - \lambda$ in a $2 - (v, k, \lambda)$ design we will write n for $r - \lambda$ and call n the *order* of the design. A further result on intersection numbers of 2-designs is the following result due to Majumdar [30].

RESULT 1.5. *Let D be a $2 - (v, k, \lambda)$ design. If x, y are two blocks of D , then $|x \cap y| \geq k - n$ with equality if and only if $|x \cap z| = |y \cap z|$ for every block z ($z \neq x$ or y).*

Note that, for most 2-designs, $k - n < 0$. In fact one can easily show that $k - n \geq 0$ if and only if $0 \leq b - v \leq r - 1$.

An *incidence matrix* $A = (a_{ij})$ for a design D , is a v by b matrix with its rows indexed by the points of D and its columns indexed by the blocks of D , such that $a_{ij} = 1$ if the i th point is incident with the j th block and $a_{ij} = 0$ otherwise.

For any design D we define the *dual* D^* of D to be the incidence structure whose points are the blocks of D and whose blocks are the points of D with incidence the “same” as in D . Clearly if D is a $1 - (v, k, r)$ design then D^* must be a $1 - (b, r, k)$ design. Furthermore we see from Result 1.4 that the dual of a symmetric $2 - (v, k, \lambda)$ design is itself a symmetric $2 - (v, k, \lambda)$ design.

The *complement* $C(D)$ of a design D has as points the points of D , blocks the blocks of D , with a point P incident with a block x in $C(D)$ if and only if P is not incident with x in D (i.e., considered as point sets, the blocks of $C(D)$ are the complements of the blocks of D).

RESULT 1.6. *If D is a $2 - (v, k, \lambda)$ design ($2 \leq k \leq v - 2$), then $C(D)$ is a $2 - (v, v - k, b - 2r + \lambda)$ design.*

Note that Results 1.1–1.4 and 1.6 may be found in [20].

The *residual* D^x of a $2 - (v, k, \lambda)$ design D (x a block of D) has as its points those points of D not incident with x , and as its blocks all the blocks of D except x , with incidence as in D . Using Result 1.4 we have the following.

RESULT 1.7. *If D is a symmetric $2 - (v, k, \lambda)$ design with $v \geq 2k$ then D^x is a $2 - (v - k, k - \lambda, \lambda)$ design for every block x of D .*

Every residual design of a symmetric 2-design has $k - n = 0$ and, if a 2-design satisfies $k = n$, then it is said to be *quasi-residual*. It is an interesting problem to decide precisely when a quasi-residual design is actually a residual.

A *parallelism* of a design D is an equivalence relation \parallel on the blocks of D such that, for any point P and any block x , there exists a unique block y incident with P such that $y \parallel x$. Bose [11] showed that if a $2 - (v, k, \lambda)$ design admits a parallelism, then $b - v \geq r - 1$, and if $b - v = r - 1$, then the design is said to be *affine*. Note that, for any two nonparallel blocks x, y in an affine design, $|x \cap y|$ is a constant.

If D is a symmetric $2 - (v, k, 1)$ design then it is a *projective plane* and a residual design of D (which is always affine) is called an *affine plane*. Furthermore (see, for instance, [20]) every quasi-residual 2-design with $\lambda = 1$ is residual and hence is an affine plane. For a summary of known results on affine designs see Shrikhande [40], and for details on projective planes see [23].

A *division* of a $1 - (v, k, r)$ design D is a partition P_1, \dots, P_{v_1} of the points of D ($v_1 < v$) such that the connection number of two distinct points is a constant λ or λ' depending only on whether the points are from the same class or not. It is normal to assume $\lambda \neq \lambda'$, otherwise D is a 2-design. With this assumption it can be shown that a division of a 1-design (if it exists) is unique, and that there exists a constant l such that $|P_i| = l$ for every i . A 1-design admitting a division is called *divisible* (note that these designs are often referred to as *group divisible* designs).

RESULT 1.8 (Bose–Conner Theorem). *Let D be a divisible $1 - (v, k, r)$ design*

with connection numbers λ and λ' and v_1 point classes, and let A be an incidence matrix for D . Then either

(i) $rk = v\lambda'$ and $b \geq \text{rank } AA^T = v - v_1 + 1$ (and D is called singular divisible);
or

(ii) $rk > v\lambda'$ and $b \geq \text{rank } AA^T = v$ (and D is called nonsingular divisible).

As in the 2-design case, if a divisible $1 - (v, k, r)$ design satisfies $b = v$ then it is called *symmetric*. For further details, see, for instance, [20, pp. 286–289].

2. Matrix decompositions

In this section, we prove some elementary results about matrix decompositions. These were first established by Block [10], and have some surprisingly strong implications when applied to incidence matrices of designs. Throughout this paper we denote the rank of a matrix X by $\rho(X)$.

If A is a v by b matrix, then a *decomposition* of A is any partition P_1, \dots, P_{v_1} of the rows of A and a partition x_1, \dots, x_{b_1} of the columns of A . If $|P_i| = l_i$ and $|x_j| = m_j$, then the l_i by m_j matrices M_{ij} (which consist of all the entries in the rows of P_i and the columns of x_j) are the *decomposition matrices* of the decomposition. Clearly an arbitrary decomposition is much too general a concept to be useful. If, for each i and j , the sum of the entries of each row of M_{ij} is a constant, which we will denote by r_{ij} , then we say that the decomposition is *row-tactical*. Similarly we say that the decomposition is *column-tactical* if the sum of each column of M_{ij} is a constant c_{ij} , and that it is *tactical* if it is both row- and column-tactical. If a decomposition is row-tactical then we define the v_1 by b_1 matrix of the row sums, R , by $R = (r_{ij})$ and if it is column-tactical we define $C = (c_{ij})$.

THEOREM 2.1. *Suppose that A is a v by b matrix with a decomposition P_1, \dots, P_{v_1} of the rows and x_1, \dots, x_{b_1} of the columns.*

(a) *If the decomposition is row-tactical, then $\rho(A) - \rho(R) \leq b - b_1$.*

(b) *If the decomposition is column-tactical, then $\rho(A) - \rho(C) \leq v - v_1$.*

Proof. We prove (b). Suppose the decomposition is column-tactical. If R is a set of $\rho(A)$ linearly independent rows of A , then the remaining $v - \rho(A)$ rows of A fall in at most $v - \rho(A)$ different row classes P_i . Thus at least $v_1 - (v - \rho(A))$ of the row classes of the decomposition are completely contained in the rows of R .

Of course $v_1 - (v - \rho(A))$ might be negative or zero but, if this is the case, then $v_1 - (v - \rho(A)) \leq 0 \leq \rho(C)$, which gives $\rho(A) - \rho(C) \leq v - v_1$, as required. So suppose $v_1 - (v - \rho(A))$ is positive and consider the corresponding $v_1 - (v - \rho(A))$ rows of C . If some linear combination of these rows was equal to zero, then the same linear combination of the rows of the corresponding $v_1 - (v - \rho(A))$ row classes of A (with each row inside a given class having the same coefficient), would

also be zero. This is an immediate consequence of the fact that the decomposition is column-tactical, since multiplying each row of M_{ij} by a constant h and adding the rows gives a row vector with each entry equal to hc_{ij} . However, the $v_1 - (v - \rho(A))$ point classes are completely contained in \mathbf{R} , and the rows of \mathbf{R} were chosen to be linearly independent. Thus the $v_1 - (v - \rho(A))$ rows of C must be linearly independent, which gives $v_1 - (v - \rho(A)) \leq \rho(C)$ or $\rho(A) - \rho(C) \leq v - v_1$. This establishes (b).

The proof of (a) is given by similar considerations on the columns of A . \square

COROLLARY 2.2. *Suppose $\rho(A) = v$.*

(a) *If the decomposition of Theorem 2.1 is row-tactical, then $b_1 - v_1 \leq b - v$.*

(b) *If the decomposition of Theorem 2.1 is column-tactical, then $0 \leq b_1 - v_1$.*

Proof. (a) Clearly, since R is a v_1 by b_1 matrix, $\rho(R) \leq v_1$. So, from Theorem 2.1(a), $v = \rho(A) \leq \rho(R) + b - b_1 \leq v_1 + b - b_1$ and hence $b_1 - v_1 \leq b - v$.

(b) Also, by Theorem 2.1(b), $v = \rho(A) \leq \rho(C) + v - v_1 \leq b_1 + v - v_1$ or $b_1 - b_1 \geq 0$. \square

3. Decompositions of designs

If A is an incidence matrix of a design \mathbf{D} , any decomposition of A gives rise to a partitioning of the points and blocks of \mathbf{D} . Similarly, any partitioning of the points and blocks of \mathbf{D} gives, in a natural way, a decomposition of A . A partitioning of the points and blocks of \mathbf{D} is called a *point-tactical*, *block-tactical* or *tactical* decomposition of \mathbf{D} if it is equivalent to a row-tactical, column-tactical or tactical decomposition of A . We can, in fact, recognise the various decompositions of \mathbf{D} without considering incidence matrices.

LEMMA 3.1. *Let P_1, \dots, P_{v_1} and x_1, \dots, x_{b_1} be a partitioning of the points and blocks of a design \mathbf{D} . Then*

(a) *It is point-tactical if and only if, for any i and j , each point of class i is incident with a constant number of blocks from class j . (We denote this constant by γ_{ij} .)*

(b) *It is block-tactical if and only if, for any i and j , each block in class j is incident with a constant number of points from class i . (We denote this constant by β_{ij} .)*

Proof. We shall prove (b). By definition the decomposition of \mathbf{D} is block-tactical if and only if it is equivalent to a column-tactical decomposition of A , i.e., if and only if, for any i and j , the entries in any column of M_{ij} add up to a constant c_{ij} . But, since A is a $(0, 1)$ -matrix, this is the same as saying that each column of M_{ij} has c_{ij} nonzero entries. However, the number of nonzero entries in a given column of M_{ij} is equal to the number of points of class i incident with that particular block of class j . This establishes (b) and a similar argument will prove (a). \square

Tactical decompositions of designs were first studied by Dembowski [19]. Their importance is illustrated by the following lemma.

LEMMA 3.2. *Let Γ be an automorphism group of a design D . Then the point and block orbits of Γ form a tactical decomposition of D .*

Proof. Let P_1, \dots, P_{v_i} be the point orbits of Γ and let x_1, \dots, x_{b_i} be the block orbits. If $|P_i| = 1$ then, trivially, for every j every point of P_i is on the same number of blocks in x_j . If $|P_i| \geq 2$, let X, Y be two distinct points in P_i and let a_1, \dots, a_m be the blocks of x_j through X . Since X and Y are in the same orbit of Γ , there exists γ in Γ with $X^\gamma = Y$, and then $a_1^\gamma, \dots, a_m^\gamma$ are the blocks of x_j through Y . Thus each point of P_i is on m blocks of x_j and, by Lemma 3.1, the decomposition is point-tactical. A similar argument shows it is also block-tactical and proves the result. \square

Now that we know that the orbits of an automorphism group form a tactical decomposition, we can use our results on decompositions to obtain an orbit theorem for designs.

THEOREM 3.3. *Let D be a $2 - (v, k, \lambda)$ design. If $\Gamma \subseteq \text{Aut } D$ has v_1 point orbits and b_1 block orbits then $0 \leq b_1 - v_1 \leq b - v$.*

Proof. If A is any incidence matrix of D , then A has rank v (see, for example, [20 p. 20]). Since the point and block orbits of Γ give a tactical decomposition of D , Theorem 3.3 is an immediate consequence of Corollary 2.2. \square

Note that Theorem 3.3 can be strengthened, since all the proof requires is for the incidence matrix A to have rank v , i.e., we only require that D be a structure with v points, b blocks and incidence matrix of rank v .

Putting $v = b$ in Theorem 3.3 gives the well known orbit theorem for symmetric designs (see, for instance, [20, p. 78]). Also, putting $b_1 = 1$, we see that, if Γ is transitive on blocks, then it is also transitive on points. The converse, of course, is not true, an easy counterexample being given by the translation group of an affine translation plane (see [23]).

It is also worth pointing out that there are many examples of tactical decompositions of designs which are not the orbits of any automorphism group. Any 1-design D has a tactical decomposition with just one point and block class. However, there are many 1-designs whose full group is not transitive on points (or blocks), for example, the automorphism group of any projective nondesarguesian translation plane (see [23]).

4. Tactical decompositions of designs

Let D be a $1 - (v, k, r)$ design, and let P_1, \dots, P_{v_i} and x_1, \dots, x_{b_i} be a tactical

LEMMA 4.2.

$$A_1 A_1^T = \begin{array}{c} \begin{array}{c} v \\ v \\ v \\ \vdots \\ v \end{array} \begin{array}{c} v \\ B \\ \vdots \\ v \end{array} \begin{array}{c} b_1 \\ \left. \begin{array}{c} \gamma_{11} \ \gamma_{12} \ \cdots \ \gamma_{1b_1} \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ \gamma_{11} \ \gamma_{12} \ \cdots \ \gamma_{1b_1} \end{array} \right\} l_1 \\ \left. \begin{array}{c} \gamma_{21} \ \gamma_{22} \ \cdots \ \gamma_{2b_1} \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ \gamma_{21} \ \gamma_{22} \ \cdots \ \gamma_{2b_1} \end{array} \right\} l_2 \\ \vdots \\ \left. \begin{array}{c} \gamma_{v_1 1} \ \gamma_{v_1 2} \ \cdots \ \gamma_{v_1 b_1} \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ \gamma_{v_1 1} \ \gamma_{v_1 2} \ \cdots \ \gamma_{v_1 b_1} \end{array} \right\} l_{v_1} \end{array} \\ \begin{array}{c} b_1 \\ \left. \begin{array}{c} \gamma_{11} \ \cdots \ \gamma_{11} \ \gamma_{21} \ \cdots \ \gamma_{21} \\ \gamma_{12} \ \cdots \ \gamma_{12} \ \gamma_{22} \ \cdots \ \gamma_{22} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \gamma_{1b_1} \ \cdots \ \gamma_{1b_1} \ \gamma_{2b_1} \ \cdots \ \gamma_{2b_1} \end{array} \right\} \begin{array}{c} l_1 \\ l_2 \\ \vdots \\ l_{v_1} \end{array} \end{array} \end{array} \begin{array}{c} m_1 \\ m_2 \quad 0 \\ \vdots \\ 0 \quad \vdots \\ \vdots \quad \vdots \\ m_{b_1} \end{array} \end{array}$$

where

$$B = nI_v + \lambda J_v + \begin{array}{c} \begin{array}{c} v \\ v \\ v \\ \vdots \\ v \end{array} \begin{array}{c} J_{l_1} \\ \vdots \\ J_{l_2} \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ J_{l_{v_1}} \end{array} \begin{array}{c} v \\ 0 \\ 0 \\ \vdots \\ 0 \end{array} \end{array}$$

where A_1^T denotes the transpose of A_1 , I_v is the v by v identity matrix, J_s is the s by s matrix all of whose entries are $+1$ and $n = r - \lambda$.

Proof. Most of the computation is straightforward. The upper left-hand corner needs the basic fact that $AA^T = nI_v + \lambda J_v$ (see [20, p. 20]). The entry in the i th row, $1 \leq i \leq v$, and $(v + j)$ th column, $1 \leq j \leq b_1$, is the inner product of the i th and $(v + j)$ th rows of A_1 . Since the $(v + j)$ th row of A_1 has $+1$ entries only under the blocks of x_j , the inner product gives the number of blocks of x_j through the i th point. Thus it is γ_{kj} , where the i th point is in P_k . \square

When we try to compute $A_1^T A_1$, the b by b matrix in the top left-hand corner involves $A^T A$. Since, in general, we do not know $A^T A$, we cannot get such a precise formulation for $A_1^T A_1$. However, we do have the following.

LEMMA 4.3.

$$A_1^T A_1 = \begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c} \beta_{11} \cdots \beta_{11} \quad \beta_{12} \cdots \beta_{12} \quad \beta_{1b_1} \cdots \beta_{1b_1} \end{array} \\ \beta_{21} \cdots \beta_{21} \quad \beta_{22} \cdots \beta_{22} \quad \beta_{2b_1} \cdots \beta_{2b_1} \\ \vdots \\ \beta_{v_1 1} \cdots \beta_{v_1 1} \quad \beta_{v_1 2} \cdots \beta_{v_1 2} \quad \beta_{v_1 b_1} \cdots \beta_{v_1 b_1} \end{array} \\ \underbrace{\hspace{2em}}_{m_1} \quad \underbrace{\hspace{2em}}_{m_2} \quad \underbrace{\hspace{2em}}_{m_{b_1}} \end{array} \\ \begin{array}{c} \begin{array}{c} \beta_{11} \beta_{21} \cdots \beta_{v_1 1} \\ \vdots \\ \beta_{12} \beta_{22} \cdots \beta_{v_1 2} \\ \vdots \\ \beta_{1b_1} \beta_{2b_1} \cdots \beta_{v_1 b_1} \\ \vdots \\ \beta_{1b_1} \beta_{2b_1} \cdots \beta_{v_1 b_1} \end{array} \\ \begin{array}{c} l_1 \\ l_2 \quad 0 \\ \vdots \\ 0 \quad \vdots \\ l_{v_1} \end{array} \end{array} \end{array} \left. \begin{array}{l} \\ \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} m_1 \\ m_2 \\ \vdots \\ m_{b_1} \end{array} \end{array}$$

where C involves $A^T A$. (We know the diagonal entries are $k + 1$.)

We now use the elementary fact that $A_1(A_1^T A_1) = (A_1 A_1^T)A_1$ to obtain Dembowski's relations.

THEOREM 4.4. (i) $\beta_{ij}m_j = \gamma_{ij}l_i$ for all i and j ;
 (ii) $\sum_{j=1}^{b_1} \gamma_{ij}\beta_{ij} = \lambda l_i + n\delta_{ih}$.

Proof. (i) Consider the $(v + j, b + i)$ th entry of both sides of $(A_1 A_1^T)A_1 = A_1(A_1^T A_1)$.

(ii) Consider the $(m, b + i)$ th entry where the m th point in our ordering belongs to P_h . \square

It should be noted that Dembowski's counting arguments are no more difficult than the ones given by the matrix approach. However, this matrix method gives still more information. Using the results of this section it is fairly straightforward to show that:

$$\det A_1 A_1^T = n^{v-v_1} \prod_{i=1}^{v_1} l_i \prod_{j=1}^{b_1} m_j.$$

Thus $A_1 A_1^T$ is nonsingular, i.e., $\rho(A_1 A_1^T) = v + b_1$. But clearly $\rho(A_1) \geq \rho(A_1 A_1^T)$, and so, since A_1 has $v + b_1$ rows, $\rho(A_1) = v + b_1$. This gives an alternative proof of $b - v \geq b_1 - v_1$, since A_1 has $b + v_1$ columns, and also shows that, if $v + b_1 = b + v_1$, then A_1 is nonsingular. This fact will be very useful in Sections 5, 8 and 9 below.

5. Strong tactical decompositions

We first state a theorem due to Block [9].

THEOREM 5.1. Any tactical decomposition of a $2 - (v, k, \lambda)$ design D satisfies $b - v \geq b_1 - v_1 \geq 0$.

Proof. The proof is identical to that of Theorem 3.3, which is just a special case of Theorem 5.1. \square

Hence it is natural to look at decompositions for which $b - v = b_1 - v_1$. Note that, if $b = v$, then $b_1 = v_1$, a result originally due to Dembowski [19], but if $b_1 - v_1 = 0$, then b does not necessarily equal v ; a counterexample is, for instance, the decomposition with only one point and one block class.

As we mentioned above, any 1-design admits the tactical decomposition in

which every point and block class has length 1. Clearly, this satisfies the equality $b - v = b_1 - v_1$, but in an uninteresting way. These tactical decompositions are called *trivial* and from now on we will restrict all discussions to nontrivial tactical decompositions.

Any tactical decomposition satisfying $b - v = b_1 - v_1$ is called *strong*, and a design admitting a strong tactical decomposition is said to be *strongly decomposable*.

If a 2-design D admits a parallelism, then a tactical decomposition of D can be obtained by taking the parallel classes for block classes and taking all the points to form a single point class. Clearly $b_1 = r$, $\beta_{1j} = 1$ and $\gamma_{1j} = k$ for $1 \leq j \leq r$. Theorem 5.1 in this case becomes $b - v \geq r - 1$, which is Bose's inequality [11]. But Bose also showed that $b - v = r - 1$ if and only if D is affine, and thus every affine design is strongly decomposable. Clearly every symmetric design is also strongly decomposable. As we shall see later, there are many strongly decomposable 2-designs which are neither affine nor symmetric.

The following theorem due to Beker [4] generalises Bose's result.

THEOREM 5.2. *If $T(D)$ is a tactical decomposition of a $2 - (v, k, \lambda)$ design D , then the following are equivalent:*

- (i) *$T(D)$ is strong.*
- (ii) *The number of points in the intersection of two distinct blocks depends only on the block classes to which they belong.*
- (iii) *Every two distinct blocks in the same block class intersect in $k - n$ points.*

The equivalence of (ii) and (iii) is immediate from Result 1.5, and the fact that (i) and (iii) are equivalent may be obtained by evaluating the sum $\sum_{i=1}^{v_1} \sum_{j=1}^{b_1} \beta_{ij} \gamma_{ij}$ in two ways. Alternatively, the theorem may be obtained by using the nonsingularity of the matrix A_1 of Section 4 when $T(D)$ is strong, and by evaluating the matrix product $A_1 A_1^T A_1$ in two ways (see [2], [5]), or by the linear algebra techniques of Bridges [14]. Using Theorem 5.2 and Result 1.5 we have the following immediate corollary.

COROLLARY 5.3. *If $T(D)$ is a tactical decomposition of a $2 - (v, k, \lambda)$ design D with the property that every two distinct blocks of any block class intersect in a constant number of points ρ , then $\rho \geq k - n$ with equality if and only if $T(D)$ is strong.*

An example of a design satisfying the conditions of the corollary with $\rho > k - r + \lambda$ may be found in [4]. The following two results appear in the same paper.

THEOREM 5.4. *If a 2-design D is strongly decomposable, then $C(D)$ is also.*

THEOREM 5.5. *The only 3-designs admitting strong tactical decompositions are Hadamard 3-designs (i.e., $3 - (4\lambda + 4, 2\lambda + 2, \lambda)$ designs).*

Theorem 5.5 generalises an earlier result due to Kimberley [29, Theorem 2], who showed that Hadamard 3-designs are the only affine 3-designs.

Finally, we give some restrictions on the parameters of a strongly decomposable 2-design. For proofs see [2].

THEOREM 5.6. *If D is a strongly decomposable 2-design then:*

- (i) $k \mid (k - n)(\lambda - k + n)$;
- (ii) $\lambda \mid (k - 1)n$;
- (iii) D is a $2 - ((n(k - 1)/\lambda) + k, k, \lambda)$ design.

THEOREM 5.7. *If B_r and B_s are block classes of a strong tactical decomposition of D , then $\rho_{rs} \leq 2n/(m_r + m_s) + k - n$, where ρ_{rs} is the intersection number of a block of class B_r and a block of class B_s .*

Note that Theorem 5.7 together with Result 1.5 gives both an upper and a lower bound on the intersection numbers of 2-designs admitting strong tactical decompositions. This result can be deduced from the work of Connor [18] and Majumdar [30] for the case $m_r = m_s = 1$.

By applying the Hasse–Minkowski theory of rational congruence to our matrix equation $A_1 A_1^T = K$ given by Lemma 4.2, we obtain the following results. (We assume throughout that $T(D)$ is a regular strong tactical decomposition of a $2 - (v, k, \lambda)$ design D , where regular implies that $l_i = l$ and $m_i = m$ for every i , $1 \leq i \leq v_1$, and for every j , $1 \leq j \leq b_1$.)

THEOREM 5.8. *If v is even and b is odd then*

- (i) m is a square.
- (ii) *If $v_1 \equiv 2 \pmod{4}$, then $n^{l-1}lx^2 - y^2 = z^2$ has solutions in integers x, y, z not all zero.*

(We could alternatively write (ii) as

- (ii) *If $v_1 \equiv 2 \pmod{4}$, then D does not exist if the square free part of $n^{l-1}l$ contains a prime $p \equiv 3 \pmod{4}$.*

In the following theorems we shall state only one of the alternatives.)

THEOREM 5.9. *If v and b are both even then, either,*

- (a) b_1 and v_1 are both odd and nlm is a square and either

- (i) $b \equiv 0 \pmod{4}$ and $v \equiv 0 \pmod{4}$, or
- (ii) $b \equiv 0 \pmod{4}$ and $v \equiv 2 \pmod{4}$ and $lx^2 - y^2 = z^2$ has solutions in integers x, y, z not all zero, or
- (iii) $b \equiv 2 \pmod{4}$ and $v \equiv 2 \pmod{4}$ and $nx^2 - y^2 = z^2$ has solutions in integers x, y, z not all zero, or
- (iv) $b \equiv 2 \pmod{4}$ and $v \equiv 0 \pmod{4}$ and $mx^2 - y^2 = z^2$ has solutions in integers x, y, z not all zero, or
- (b) b_1 and v_1 are both even and either
 - (i) $b \equiv 0 \pmod{4}$ and $v \equiv 2 \pmod{4}$ and $v_1 \equiv 2 \pmod{4}$ and $b_1 \equiv 0 \pmod{4}$ and $lx^2 - y^2 = z^2$ has solutions in integers x, y, z not all zero, or
 - (ii) $b \equiv 0 \pmod{4}$ and $v \equiv 0 \pmod{4}$ and either $b_1 \equiv v_1 \equiv 0 \pmod{4}$ or $b_1 \equiv v_1 \equiv 2 \pmod{4}$ and $mlnx^2 - y^2 = z^2$ has solutions in integers x, y, z not all zero, or
 - (iii) $b \equiv 2 \pmod{4}$, $v \equiv 2 \pmod{4}$ and $b_1 \equiv v_1 \equiv 2 \pmod{4}$ and $mlx^2 - y^2 = z^2$ has solutions in integers x, y, z not all zero, or
 - (iv) $b \equiv 2 \pmod{4}$, $v \equiv 0 \pmod{4}$ and $v_1 \equiv 0 \pmod{4}$, $b_1 \equiv 2 \pmod{4}$ and $mx^2 - y^2 = z^2$ has solutions in integers x, y, z not all zero.

THEOREM 5.10. *If v and b are both odd then lm is a square and either*

- (i) $b \equiv 1 \pmod{4}$ and $v \equiv 1 \pmod{4}$ and either $b_1 \equiv v_1 \equiv 1 \pmod{4}$ and $nx^2 + ly^2 = z^2$ has solutions in integers x, y, z not all zero or $b_1 \equiv v_1 \equiv 3 \pmod{4}$ and $nx^2 - ly^2 = z^2$ has a solution in integers x, y, z not all zero, or
- (ii) $b \equiv 1 \pmod{4}$ and $v \equiv 3 \pmod{4}$ and either $b_1 \equiv 1 \pmod{4}$, $v_1 \equiv 3 \pmod{4}$ and $mx^2 - ny^2 = z^2$ has solutions in integers x, y, z not all zero or $v_1 \equiv 1 \pmod{4}$ and $b_1 \equiv 3 \pmod{4}$ and $(-m, -n)_p = +1$ for all odd primes p , or
- (iii) $b \equiv 3 \pmod{4}$ and $v \equiv 1 \pmod{4}$ and either $b_1 \equiv 1 \pmod{4}$ and $v_1 \equiv 3 \pmod{4}$ and $(-m, -n)_p = +1$ for all odd primes p or $v_1 \equiv 1 \pmod{4}$, $b_1 \equiv 3 \pmod{4}$ and $mx^2 - ny^2 = z^2$ has solutions in integers x, y, z not all zero, or
- (iv) $b \equiv 3 \pmod{4}$ and $v \equiv 3 \pmod{4}$ and either $b_1 \equiv v_1 \equiv 1 \pmod{4}$ and $nx^2 - my^2 = z^2$ has solutions in integers x, y, z not all zero, or $b_1 \equiv v_1 \equiv 3 \pmod{4}$ and $nx^2 + my^2 = z^2$ has solutions in integers x, y, z not all zero.

THEOREM 5.11. *If v is odd and b is even then*

- (1) l must be a square,
- (2) either $b \equiv 0 \pmod{4}$ and $nm x^2 + (-1)^{v-d/2} y^2 = z^2$ has solutions in integers x, y, z not all zero or $b \equiv 2 \pmod{4}$ and $nx^2 + (-1)^{v-d/2} y^2 = z^2$ has solutions in integers x, y, z not all zero.

In the case when D is symmetric or affine these results reduce to the results of Bruck–Ryser–Chowla [15], [17] and Shrikhande [37], [38].

6. Resolutions

In this section we consider 2-designs which admit a special type of tactical decomposition called a resolution. A *resolution* $R(\mathbf{D})$ of a $1 - (v, k, r)$ design \mathbf{D} is a tactical decomposition with just one point class, and a design admitting a resolution is called *resolvable*.

Note that this definition is not standard, in fact many authors (see, for instance, [40]) use the word to describe what we call a parallelism. The definition used here corresponds with the definition of $(\mu_1, \mu_2, \dots, \mu_r)$ -resolvable due to Kageyama [26], which generalised the idea of α -resolvability initiated by Shrikhande and Raghavarao [41].

We first give the following theorem due to Beker [2].

THEOREM 6.1. *If \mathbf{D} is a resolvable $1 - (v, k, r)$ design with the property that every two distinct blocks of the same block class have a constant intersection number, ρ , say, then :*

- (i) *any point is incident with the same number $\sigma = (k^2 - k\rho)/(k^2 - v\rho)$ of blocks from each block class ;*
- (ii) *each block class has the same number $m = (vk - v\rho)/(k^2 - v\rho)$ of blocks ;*
- (iii) *$\rho \leq k(k - 1)/(v - 1)$.*

By Theorem 5.1, any resolution of a 2-design satisfies $b - v \geq b_1 - 1$ and we accordingly define a *strong resolution* to be one for which $b = v + b_1 - 1$, and we call a 2-design admitting a strong resolution *strongly resolvable*. By definition, we see that a strongly resolvable 2-design is simply a special type of strongly decomposable 2-design, and the following theorem due to Hughes and Piper [24] is an immediate corollary of Theorem 5.2.

THEOREM 6.2. *If $R(\mathbf{D})$ is a resolution of a $2 - (v, k, \lambda)$ design \mathbf{D} , then the following are equivalent :*

- (i) *$R(\mathbf{D})$ is strong.*
- (ii) *The intersection number of two blocks depends only on the block classes to which they belong.*
- (iii) *Every two distinct blocks of the same block class have intersection number $k - n$.*

In fact, Hughes and Piper showed that, in this case, we have the stronger result that the intersection number of two blocks from different classes must also be a constant equal to k^2/v . Thus, a strongly resolvable 2-design is precisely what Shrikhande and Raghavarao [41] call an affine α -resolvable design. This paper

contains a weaker version of Theorem 6.2; it includes the added assumption that the resolution is regular (a *regular* resolution is one for which each block class contains the same number of blocks). Clearly, by Theorem 6.1, a strong resolution is always regular. Other authors, e.g., Shah [36], have given further necessary and sufficient conditions for a regular resolution to be strong.

We can now state a theorem due to Beker and Haemers [6].

THEOREM 6.3. *If D is a $2 - (v, k, \lambda)$ design then the following are equivalent:*

- (i) *D is strongly resolvable.*
- (ii) *D has precisely two intersection numbers, the smaller of which is $k - n$.*
- (iii) *D has precisely two intersection numbers, the larger of which is k^2/v .*
- (iv) *D^* is divisible.*
- (v) *D^* is singular divisible.*

We also have the following result on the parameters of strongly resolvable 2-designs; see [2] and [21].

THEOREM 6.4. *If D is a strongly resolvable 2-design then:*

- (i) *D is a $2 - (\mu m^2/\sigma^2, \mu m/\sigma, (\mu m - \sigma)/(m - 1))$ design, where m is the block class size, μ is the intersection number of two blocks from different block classes of the resolution, and $\sigma = \gamma_{1j}$ for every j .*
- (ii) *$\sigma = nk/(k^2 - kv + nv)$; $m = nv/(k^2 - kv + nv)$.*

Clearly, strongly resolvable designs include affine designs as a special case (i.e., when $k - n = 0$), but there do exist infinite families of strongly resolvable 2-designs which are not affine, see Section 9 below.

7. 2-Designs with $k - n$ as an intersection number

We now consider $2 - (v, k, \lambda)$ designs having $k - n$ as an intersection number, and so, by Theorem 5.2, we consider all strongly decomposable 2-designs.

We first give the following constraints on the parameters of such a design (for proofs see [6]).

THEOREM 7.1. *If a $2 - (v, k, \lambda)$ design contains two blocks having intersection number $k - n$ then:*

- (i) $0 \leq k - n \leq \lambda$;
- (ii) $v - k \geq n$;
- (iii) $v - 1 \geq b - r$;

- (iv) $v - 1 \geq r$;
- (v) $k \geq r/2$.

There exist infinitely many affine and symmetric 2-designs and so there exist infinitely many 2-designs with intersection number $k - n$ whenever $k - n = 0$ or λ . However, as shown in [6],

THEOREM 7.2. *For a given value of $k - n \notin \{0, \lambda\}$, there exist only finitely many 2-designs admitting $k - n$ as an intersection number.*

We also have the following.

THEOREM 7.3. *For a given value of λ , there exist only finitely many 2-designs with an intersection number $k - n \notin \{0, \lambda\}$.*

Note that the total number of 2-designs admitting an intersection number $k - n \neq 0, \lambda$ is not finite. In Section 10 we will construct an infinite family of such designs.

Beker and Haemers also point out that the only 3-designs admitting an intersection number $k - n$ are the Hadamard 3-designs, giving a further generalisation of Theorem 5.5.

We now define a relation on the blocks of a 2-design \mathbf{D} admitting $k - n$ as an intersection number. If x, y are blocks of \mathbf{D} , then $x \sim y$ if and only if $x = y$ or $|x \cap y| = k - n$. By Result 1.5, \sim is an equivalence relation on the blocks of \mathbf{D} , and for the purposes of this section we will call the partition of the blocks of \mathbf{D} into equivalence classes under \sim the *maximal $(k - n)$ -decomposition* of \mathbf{D} . Any refinement of the maximal $(k - n)$ -decomposition is called a $(k - n)$ -decomposition of the design. A $(k - n)$ -decomposition of a 2-design has the property that the intersection number of two blocks depends only on their decomposition classes. An example of a $(k - n)$ -decomposition is provided by the block classes of a strong tactical decomposition of a 2-design.

A decomposition is said to be *regular* if each class contains the same number of blocks, and an example of a regular maximal $(k - n)$ -decomposition is provided by the block classes of any strong resolution of a 2-design. The relationship between $(k - n)$ -decompositions and resolutions is strengthened by the following result, see [6] and [31].

THEOREM 7.4. *If $\mathbf{B}_1, \dots, \mathbf{B}_b$ are the classes of a $(k - n)$ -decomposition of a $2 - (v, k, \lambda)$ design then,*

- (a) $|B_j| \leq b/(b - v + 1)$, for every j .
- (b) $v + b_1 \geq b + 1$.
- (c) *The following are equivalent:*
 - (i) B_1, \dots, B_{b_1} is a strong resolution;
 - (ii) $|B_j| = b/(b - v + 1)$ for every j ;
 - (iii) $v + b_1 = b + 1$.

Restricting our attention to regular $(k - n)$ -decompositions (and writing m for the size of a class in such a decomposition) we have the following results (see [6]).

THEOREM 7.5. *If D is a nonsymmetric $2 - (v, k, \lambda)$ design with a regular $(k - n)$ -decomposition, then:*

$$k - n \leq \lambda v/b \leq n/m + k - n.$$

THEOREM 7.6. *Let D be a $2 - (v, k, \lambda)$ design with a regular $(k - n)$ -decomposition and suppose ρ_{ij} is the intersection number of a block of class B_i and a block of class B_j ($i \neq j$). Then:*

- (i) $\rho_{ij} \geq 2k^2/v - n/m - k + n$, with equality if and only if all the points of D are in the same number of blocks from $B_i \cup B_j$;
- (ii) $\rho_{ij} \leq n/m + k - n$ with equality if and only if each point occurs the same number of times in blocks from B_i as in blocks from B_j .

Note that (ii) of the above result generalises Theorem 5.7 on intersection numbers between classes of a strong tactical decomposition; it is also worth pointing out that in the case when D is a strongly resolvable 2-design and we consider the maximal $(k - n)$ -decomposition (which is clearly regular), both the above bounds are sharp. This theorem can be deduced from the work of Connor [18] and Majumdar [30] for the case $m = 1$.

We now consider 2-designs with exactly three intersection numbers, one of which is $k - n$. In [6] the following is shown.

THEOREM 7.7. *If D is a $2 - (v, k, \lambda)$ design with exactly three intersection numbers, $k - n$, ρ_1 and ρ_2 , then the maximal $(k - n)$ -decomposition is regular, and the size m of the classes is given by:*

$$m = \frac{\lambda k^2 - kn + n^2 + b\rho_1\rho_2 - \lambda v(\rho_1 + \rho_2)}{(k - n - \rho_1)(k - n - \rho_2)}.$$

A similar result does not hold for designs with greater than three intersection numbers, see, for instance, the design of Bhattacharya [9].

We also have the following.

THEOREM 7.8. *If D is a $2-(v, k, \lambda)$ design with exactly three intersection numbers, $k - n$, ρ_1 and ρ_2 , then $\rho_1 = n/m + k - n$ if and only if $\rho_2 = \lambda v/b$.*

It does not seem likely that a 2-design with a regular maximal $(k - n)$ -decomposition need to have three or less intersection numbers, although the authors know of no examples having greater than three. As we shall see in Section 10 below, all known constructions of strongly decomposable 2-designs yield designs having three or less intersection numbers.

However, in some special cases we can restrict the number of possible block intersection sizes. Singhi and Shrikhande [44] showed:

THEOREM 7.9. *If a quasi-residual $2-(v, k, \lambda)$ design has a regular $(k - n)$ -decomposition with $\lambda - 1$ blocks in a block class, then the only possible other intersection numbers are λ and $\lambda - 1$.*

Another result of this type is found in [6].

THEOREM 7.10. *If D is a $2-(v, k, \lambda)$ design with a regular maximal $(k - n)$ -decomposition, and $\lambda v/b$ and $n/m + k - n$ are consecutive integers, then D has at most three intersection numbers.*

(If we set $h = 2$ in the family of designs of Theorem 10.1, then we obtain examples of both Theorems 7.9 and 7.10.)

Beker and Haemers [6] also gave some conditions for certain quasi-residual designs having three or less intersection numbers to be residuals of symmetric 2-designs. An application of one of their results is the embedding of the quasi-residual designs which shall be discussed in Section 10.

8. Strongly divisible 1-designs

Many of the results which hold for strongly decomposable 2-designs may be generalised to a certain special class of 1-designs. A *tactical division* of a 1-design is a tactical decomposition with point classes P_1, \dots, P_{v_1} such that:

(i) The connection number of two distinct points $P \in P_i, Q \in P_j$ depends only on the choice of i and j and is denoted by λ_{ij} .

(ii) There exists a constant λ such that $\lambda_{ii} = \lambda$ for every i ($1 \leq i \leq v_1$) with $|P_i| > 1$ (i.e., for every i such that λ_{ii} is defined).

If a 1-design D admits a tactical division, then D is said to be *tactically divisible*. Clearly, if D is a 2-design, then the terms tactical division and tactical decomposition are synonymous.

As a partial generalisation of Block's Theorem (theorem 5.7) Beker [5] showed:

THEOREM 8.1. *Any tactical division of a $1 - (v, k, r)$ design satisfies $b - v \geq b_1 - v_1$.*

Beker's proof uses the matrix A_1 , defined in an analogous way to the A_1 of Section 4. It is shown in [5] that $A_1 A_1^T$ is nonsingular, in fact $|A_1 A_1^T| = n^{v-v_1} \prod_{i=1}^{v_1} l_i \prod_{j=1}^{b_1} m_j$, where $n = r - \lambda$ and λ is as in (ii) of the definition of tactical division above. We will use n in this generalised sense throughout this section.

As in the 2-design case (see Theorem 4.4) we have the following.

THEOREM 8.2. *If $T(D)$ is a tactical division of a 1-design D , then:*

- (i) $m_j \beta_{ij} = l_i \gamma_{ij}$ for every i and j ;
- (ii) $\sum_{u=1}^{b_1} \gamma_{iu} \beta_{ju} = l_j \lambda_{ij} + n \delta_{ij}$ for every i and j .

The theorem may be obtained by considering the matrix identity $A_1 A_1^T A_1$ in two ways, precisely as in the method of proof of Theorem 4.4.

As in Section 5, we now define a *strong* tactical division to be one for which $b - v = b_1 - v_1$, and a design admitting a strong tactical division is said to be *strongly divisible*. We can obtain the following generalisation of Theorem 5.2.

THEOREM 8.3. *If $T(D)$ is a tactical division of a $1 - (v, k, r)$ design D , then the following are equivalent:*

- (i) $T(D)$ is strong.
- (ii) The intersection number of two blocks depends only on their block classes.
- (iii) The intersection number of any two distinct blocks of the same block class is $k - n$.

The original proof of this theorem again depends on the nonsingularity of A_1 . A_1^{-1} is calculated and $A_1 A_1^T A_1$ is evaluated in two ways to give the equivalence of the above three statements. Alternatively, using counting methods, Mitchell [31] obtains the result by first proving Theorem 8.4 below to obtain the equivalence of (ii) and (iii), and then evaluating the sum $\sum_{i=1}^{v_1} \sum_{j=1}^{b_1} \beta_{ij} \gamma_{ij}$ in two ways to obtain the equivalence of (i) and (iii). (This method is a direct generalisation of the technique employed in [4] to prove Theorem 5.2.) Theorem 8.1 also follows from Mitchell's proof.

As a byproduct of the original proof [5] of Theorem 8.2 one obtains the following generalisation of Corollary 5.3:

THEOREM 8.4. *If $T(\mathbf{D})$ is a tactical division of a $1 - (v, k, r)$ design \mathbf{D} , with the property that every two distinct blocks of the same block class intersect in a constant number of points ρ , then $\rho \geq k - n$, with equality if and only if $T(\mathbf{D})$ is strong.*

From Theorem 8.3 we also have the following.

COROLLARY 8.5. *A 1-design \mathbf{D} is strongly divisible iff the dual design \mathbf{D}^* is also.*

Hence strongly divisible 1-designs have many properties akin to those of symmetric 2-designs.

As an intermediate corollary of Theorem 8.2 (ii) and Corollary 8.5 we have:

COROLLARY 8.6. *If $T(\mathbf{D})$ is a strong tactical division of a 1-design \mathbf{D} then,*

$$\sum_{u=1}^{v_1} \beta_{ui} \gamma_{uj} = m_{ij} \rho_{ij} + n \delta_{ij} \quad \text{for every } i, j.$$

Evaluating the matrix product $AA^T A$ in two ways, leads to the following.

LEMMA 8.7. *If $T(\mathbf{D})$ is a strong tactical division of a 1-design \mathbf{D} , and \mathbf{B}_i is any block class satisfying $|\mathbf{B}_i| > 1$, then*

$$\sum_{u=1}^{v_1} \lambda_{ui} \beta_{uj} = \sum_{w=1}^{b_1} \rho_{jw} \gamma_{iw} \quad \text{for every } i.$$

We also have (see [5]) a generalisation of Theorem 5.4.

THEOREM 8.8. *If \mathbf{D} is a strongly divisible 1-design then $C(\mathbf{D})$ is also.*

Using more “matrix counting”, two further combinatorial identities can be established ([2]).

THEOREM 8.9. *If $T(\mathbf{D})$ is a strong tactical division of a $1 - (v, k, r)$ design, then:*

$$\sum_{j=1}^{v_1} l_j \lambda_{ij} = \sum_{s=1}^{b_1} m_s \rho_{si} = rk - n \quad \text{for every } i (1 \leq i \leq v_1) \text{ and for every } s (1 \leq s \leq b_1).$$

THEOREM 8.10. *If $T(\mathbf{D})$ is a strong tactical division of a $1 - (v, k, r)$ design, then for any two distinct block classes \mathbf{B}_s and \mathbf{B}_t :*

$$\sum_{i=1}^{v_1} \sum_{j=1}^{v_1} (\beta_{is} - \beta_{it})(\beta_{js} - \beta_{jt})\lambda_{ij} = 2n(k - n - \rho_{st}) + \sum_{h=1}^{b_1} (\rho_{sh} - \rho_{th})^2 m_h,$$

where ρ_{st} is the intersection number of a block of class \mathbf{B}_s and a block of class \mathbf{B}_t .

Theorem 8.9 can be obtained by showing that, for any tactical division, $\sum_{j=1}^{v_1} l_j \lambda_{ij} = rk - n$, and the theorem then follows by Corollary 8.5. Also (again by Corollary 8.5), we may obtain the dual result of Theorem 8.10.

Finally, using Hasse–Minkowski theory precisely as in Section 5 above, Beker [2] obtains nonexistence results for 1-designs admitting regular strong tactical divisions. These results are “identical” to those of Theorems 5.8–5.11.

9. Orbit theorems

As shown in Section 3 above, the point and block orbits of an automorphism group of a design form a tactical decomposition and, as in Theorem 3.3, we may use results obtained on tactical decompositions of designs to obtain orbit theorems. It is also possible, as we shall see below, to derive strong tactical decompositions of 2-designs from automorphism groups of these designs.

We first give an orbit theorem due to Beker ([2], [3]) for any automorphism group that permutes the point and block classes of a strong tactical division of a 1-design.

We require the following notation. Suppose $T(\mathbf{D})$ is a tactical division of a 1-design \mathbf{D} , and α is an automorphism of \mathbf{D} . Then we denote the number of fixed points of α by $\chi_x(\alpha)$, the number of fixed blocks of α by $\chi_y(\alpha)$, the number of fixed block classes of α by $\chi_{\mathbf{B}}(\alpha)$, and the number of fixed point classes of α by $\chi_{\mathbf{P}}(\alpha)$.

THEOREM 9.1. *Let $T(\mathbf{D})$ be a strong tactical division of a 1-design \mathbf{D} . If α is an automorphism of \mathbf{D} permuting the point and block classes of $T(\mathbf{D})$, then:*

$$\chi_x(\alpha) + \chi_{\mathbf{B}}(\alpha) = \chi_y(\alpha) + \chi_{\mathbf{P}}(\alpha).$$

The proof depends on the nonsingularity of A_1 , as defined in Section 4. As corollaries, we have results due to Parker [34] and Beker [3]; it is perhaps worth noting that the special case of Corollary 9.2 when $\lambda = 1$ is due to Baer [1].

COROLLARY 9.2. *If α is an automorphism of a symmetric 2-design, then it fixes the same number of points and blocks.*

COROLLARY 9.3. *If α is an automorphism of a strongly resolvable 2-design, then $\chi_x(\alpha) + \chi_B(\alpha) = \chi_y(\alpha) + 1$.*

Now suppose Π is an automorphism group of a 1-design \mathbf{D} with a strong tactical division $T(\mathbf{D})$, such that Π permutes the point and block classes of $T(\mathbf{D})$. We denote the number of point orbits by t_1 , the number of block orbits by t_2 , the number of block class orbits by t_3 , and the number of point class orbits by t_4 .

The following theorem is shown in [4].

THEOREM 9.4. *If \mathbf{D} and Π are as above, then :*

$$t_1 + t_3 = t_2 + t_4.$$

The proof uses Theorem 9.1 above. This theorem generalises the corresponding result for strongly decomposable 2-designs in [3], which itself had as corollaries the following earlier result due to Norman [33] and Harris [21].

COROLLARY 9.5. *If Π is an automorphism group of a strongly resolvable design, then $t_1 + t_3 = t_2 + 1$.*

As a further corollary we have an alternative proof of the orbit theorem for symmetric designs (see Section 3).

We now consider automorphism groups which leave decomposition classes invariant. (The results in the rest of this section are all due to Beker [2].)

LEMMA 9.6. *Let $T(\mathbf{D})$ be a strong tactical decomposition of a 2-design \mathbf{D} . If Π is an automorphism group of \mathbf{D} that leaves the block classes of $T(\mathbf{D})$ invariant, then Π leaves the point classes invariant.*

Using Lemma 9.6 we have the following.

THEOREM 9.7. *Let $T(\mathbf{D})$ be a strong tactical decomposition of a 2-design \mathbf{D} . If Π is an automorphism group of \mathbf{D} that leaves the block classes of $T(\mathbf{D})$ invariant, then the point and block orbits form a strong tactical decomposition of \mathbf{D} .*

If $T(\mathbf{D})$ is maximal (i.e., if the block classes of $T(\mathbf{D})$ form a maximal $(k - n)$ -decomposition in the sense of Section 7), then we have a stronger result:

THEOREM 9.8. *Let $T(\mathbf{D})$ be a maximal strong tactical decomposition of a 2-design \mathbf{D} , and let Π be an automorphism group of \mathbf{D} . Then the point and block orbits of Π form a strong tactical decomposition if and only if Π leaves the block classes of $T(\mathbf{D})$ invariant.*

From Theorem 9.8 we see that every automorphism of a symmetric 2-design gives rise to a strong tactical decomposition of the design, since, for a symmetric 2-design, the tactical decomposition with just one point and block class is strong.

We finally give two further results indicating methods of obtaining strong tactical decompositions of 2-designs by using automorphism groups.

THEOREM 9.9. *If \mathbf{D} is a symmetric $2-(v, k, \lambda)$ design with an axial automorphism α , and $\mathbf{D}' = \mathbf{D}^x$ where x is the axis of α , then \mathbf{D}' is a strongly decomposable 2-design.*

THEOREM 9.10. *Let \mathbf{D} be a 2-design and Π an automorphism group of \mathbf{D} such that the point and block orbits of Π form a strong decomposition of \mathbf{D} . If Γ is a subgroup of Π then the point and block orbits of Γ form a strong tactical decomposition of \mathbf{D} .*

10. Constructions and examples

We provide here a list of known families of strongly decomposable 2-designs. The construction methods used to obtain these designs may also be used to construct many strongly divisible 1-designs.

The first result uses a construction method due to Beker and Mitchell [7]. Details of this particular example of the construction may be found in [32].

THEOREM 10.1. *If there exists an affine plane of order $q - 1$ and $q > 2$ is a prime power, then for every $h \geq 2$ there exists a $2-((q - 1)(q^h - 1), q^{h-1}(q - 1), q^{h-1})$ design \mathbf{D} admitting a strong tactical decomposition with $(q^h - 1)/(q - 1)$ point classes, $q(q^h - 1)/(q - 1)$ block classes of $(q - 1)^2$ points and $(q - 1)$ blocks each, respectively. \mathbf{D} has intersection numbers $0 (= k - n)$, $q^{h-2}(q - 1)$ and q^{h-1} .*

Since the designs of Theorem 10.1 have $k - n = 0$, they are quasi-residual, and it can be shown (see for instance [6, section 6]) that they are in fact residual. Hence the existence is established of an infinite family of symmetric $2-(q^{h+1} - q + 1, q^h, q^{h-1})$ designs. The construction of Theorem 10.1 may be regarded as a generalisation of the construction method of [8].

Shrikhande and Raghavarao [42] give a method of construction of an infinite family of strongly resolvable 2-designs which are never affine. Their result may be stated by the following theorem.

THEOREM 10.2. *If there exists an affine $2-(ut^2, ut, (ut - 1)/(t - 1))$ design and a symmetric $2-(t, s, s(s - 1)/(t - 1))$ design for some s , then there exists a strongly resolvable $2-(ut^2, uts, s(uts - 1)/(t - 1))$ design D . D has parameters $b = t(ut^2 - 1)/(t - 1)$, $r = s(ut^2 - 1)/(t - 1)$, $\sigma = s$, $m = t$ and $\mu = us^2$.*

Note that, since $k - n = uts(s - 1)/(t - 1) > 0$, these designs are never affine. The family of designs of this theorem are the only strongly resolvable designs known to the authors which are not affine (apart from complements of affine 2-designs).

We now consider a construction due to Sillitto [43]. We first note the importance of a special class of 2-designs. If a $2-(v, k, \lambda)$ design satisfies $b = 4n$, then it is said to satisfy *Condition S* and we immediately have the following results due to Sillitto [43] and Shrikhande [39]:

LEMMA 10.3. *D is a $2-(v, k, \lambda)$ design satisfying Condition S if and only if D is a $2-(u^2, u(u \pm 1)/2, (u \pm 2)N/2)$ design, where u, N are integers satisfying $N \geq u/2 \geq 1$. In this case $b = 2uN$ and $r = (u \pm 1)N$. Furthermore, D is a symmetric $2-(v, k, \lambda)$ design satisfying Condition S if and only if D is a $2-(4s^2, s(2s \pm 1), s(s \pm 1))$ design.*

In the above lemma the designs with the parameters corresponding to the + signs represent the complements of the designs with parameters corresponding to the - signs. If D is a $2-(u^2, u(u - 1), (u - 2)N/2)$ design, then we say D is a $U(u, N)$ design and, if D is a $2-(4s^2, s(2s - 1), s(s - 1))$ design, then we call D an $S(s)$ design. Note that an $S(s)$ design is the same as a $U(2s, s)$ design. Also, as shown in [35], [39], [43], and [45], an $S(s)$ design exists for infinitely many values of s , and, in particular, for every s satisfying $1 \leq s \leq 10$.

Using the construction method of [43], Mitchell [31] obtained the following.

THEOREM 10.4. *Suppose there exists an $S(s)$ design and a $U(u, N)$ design admitting a strong tactical decomposition with v_1 point classes, b_1 block classes, and intersection numbers ρ_{ij} (i.e., a block of the i th class and a block of the j th class have intersection number ρ_{ij}), then there exists a $U(2us, 2Ns)$ design admitting a strong tactical decomposition with $4s^2v_1$ point classes, $4s^2b_1$ block classes, and intersection numbers $us(2s - 1) + 4s^2\rho_{ij}$, $us(us - 1)$.*

This establishes a recursive method for constructing strongly decomposable $U(u, N)$ designs.

The only affine design satisfying Condition S is the unique affine plane of order 3, namely the $U(3, 2)$. Of the strongly decomposable designs constructed in Theorems 10.1 and 10.2 above, the only designs which satisfy Condition S are as follows. Let $t \equiv 3 \pmod{4}$ ($t \geq 7$), be a prime power. Then there exists an affine $2 - (t^2, t, 1)$ design and a symmetric (Hadamard) $2 - (t, (t-1)/2, (t-3)/4)$ design (see, for instance, [20, p. 112]). Then, using Theorem 10.4, there exists a strongly resolvable $U(t, (t+1)/2)$ design with $\sigma = (t-1)/2$, $\mu = (t-1)^2/4$, $m = t$ and $k - n = t(t-3)/4$. Using these designs together with the $U(3, 2)$ above in Theorem 10.4, we obtain the following theorem.

THEOREM 10.5. *If $t \equiv 3 \pmod{4}$ and an $S(s)$ design exists, then there exists a $U(2st, s(t+1))$ design admitting a strong tactical decomposition with $4s^2$ point classes, $4s^2(t+1)$ block classes, and intersection numbers $st(st-s-1)$ (“ $k-n$ ”), $st(st-1)$ and $st(st-1) + s^2$.*

Note that designs with identical parameters were constructed, using a different method, in [7]. Furthermore, the use of two divisible designs of John and Turner [25], leads to the construction (see [7]) of strongly decomposable $U(8, 5)$ and $U(10, 6)$ designs with parameters corresponding to $m = 4, s = 1$ and $m = 5, s = 1$ in Theorem 10.5, respectively. These designs may also be used in Theorem 10.4 to generate strongly decomposable $U(16s, 10s)$ and $U(20s, 12s)$ designs for every s such that an $S(s)$ design exists.

Clearly there exist many infinite families of strongly divisible 1-designs which are not 2-designs. As we see in Theorem 11.7 below, any divisible 1-design with a divisible dual is strongly divisible, and there exist many infinite families of these designs alone. Also note that the theorems above automatically yield strongly divisible 1-designs, merely by considering the dual designs in each case.

11. Other results

There do exist further results on 1- and 2-designs admitting strong tactical divisions. In this section we outline some of these results.

We first consider tactical decompositions of symmetric 2-designs (which are necessarily strong by Theorem 5.1). The following two results may be found in [20]. Dembowski [19] showed:

THEOREM 11.1. *Let $T(D)$ be a regular tactical decomposition of a symmetric $2 - (v, k, \lambda)$ design D with v_1 point and block classes. If n is not a square, then v_1 is odd.*

Hughes [22] proved the following.

THEOREM 11.2. *Let $T(D)$ be a regular tactical decomposition of a symmetric $2-(v, k, \lambda)$ design D with v_1 point and block classes of l points and blocks each. If n is not a square, then the equation*

$$nx^2 + (-1)^{(v_1-1)/2} \lambda ly^2 = z^2$$

has solutions in integers x, y, z not all zero.

Further results of this type may be found in [20, pp. 61–63].

Secondly, we examine the relationships between divisions and tactical divisions of 1-designs established by Mitchell [31]. The following theorem generalizes an earlier result due to Shrikhande and Raghavarao [41, corollary 6.1].

THEOREM 11.3. *Let $T(D)$ be a strong tactical division of a 1-design D with point classes P_1, \dots, P_{v_1} and block classes B_1, \dots, B_{b_1} . then the following are equivalent:*

- (i) P_1, \dots, P_{v_1} is a singular division of D ;
- (ii) P_1, \dots, P_{v_1} are the classes of a resolution of D^* ;
- (iii) B_1, \dots, B_{b_1} is a singular division of D^* ;
- (iv) B_1, \dots, B_{b_1} are the classes of a resolution of D .

We also have the following.

THEOREM 11.4. *Let D be a divisible 1-design. If D^* is also divisible, then the classes of the divisions of D and D^* form a strong tactical division of D , and either:*

- (i) D, D^* are singular divisible, $\beta_{ij} = k/v_1, \gamma_{ij} = r/b_1$ for every i, j ; or
- (ii) $b = v$; D, D^* are nonsingular divisible and $\beta_{ij} = \gamma_{ij}$ for every i, j .

As immediate corollaries we have results previously established by Bose and Shrikhande [12] and [13] and Kageyama [27]:

COROLLARY 11.5. *Let D be a symmetric divisible $1-(v, k, k)$ design such that D^* is divisible with the same parameters as D . Then the classes of the divisions of D and D^* form a strong tactical division and $\beta_{ij} = \gamma_{ij}$ for every i, j .*

COROLLARY 11.6. *Let D be a divisible 1-design with a divisible dual. Then D is singular divisible if and only if D^* is singular divisible.*

Mitchell also proved the following.

THEOREM 11.7. *If D is a symmetric divisible 1-design, then the following are equivalent:*

- (i) D admits a strong tactical division with point classes the classes of the division of D ;
- (ii) D^* is divisible with the same parameters as D ;
- (iii) D^* is divisible.

Finally, we consider some further results on 2-designs having precisely three intersection numbers, one of which is $k - n$. The result given here was established by Beker and Haemers [6].

Suppose D is a $2 - (v, k, \lambda)$ design with three intersection numbers $k - n$, ρ_1 and ρ_2 ($\rho_1 > \rho_2$). Define the *class graph* G of D to be the graph whose vertices are the classes of the maximal $(k - n)$ -decomposition of D (in the sense of Section 7) and where two vertices are adjacent if and only if two blocks, one from each of the corresponding classes, have intersection number ρ_2 . Hence G has b_1 vertices, where b_1 is the number of classes in the maximal $(k - n)$ -decomposition.

THEOREM 11.8. *G is strongly regular, and G is a complete d -partite graph (for some d) if and only if $\rho_2 = \lambda v / b$.*

Beker and Haemers also calculate the eigenvalues of G , and show that D^* must be a PBIBD with three associate classes. Note that the families of designs of Theorems 10.1 and 10.5 are all examples of Theorem 11.8.

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Henry Beker,
Racal Research Ltd.,
Worton Drive,
Worton Grange Industrial Estate,
Reading, Berkshire, RG2 05B,
UK.

Christopher Mitchell,
Racal-Comsec Ltd.,
Milford Industrial Estate,
Tollgate Road,
Salisbury, Wiltshire, SP1 2JG,
UK.

Fred Piper,
Department of Maths,
Westfield College,
University of London,
Kidderpore Avenue,
London, NW3 7ST,
UK.